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COMMENT

Sequence of discrete spin models approximating the classical Heisenberg ferromagnet

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Abstract. Systematic investigation of a recent proposal of Rapaport to describe the Heisenberg model with discrete spin models is presented in two and three dimensions. It is argued that in three dimensions a 12-state discretisation already accounts satisfactorily for the critical properties, the corresponding anisotropy operators being irrelevant.

Discrete spin models are defined by replacing the continuously oriented unit-length vector variable by a set of discrete directions selected regularly on the surface of the unit sphere. A sufficiently dense discretisation is expected to have the same phase structure and very close critical characteristics to the original Heisenberg model. The physical argument in favour of this expectation is to consider the large-distance correlations in terms of block variables. For large enough blocks, these variables should be indistinguishable from the continuously orientable spins.

The gain in the increased efficiency of computer simulations resulting from fast algorithms especially devised for discrete spin models might balance the more significant finite-size corrections implied by the block-spin argument. The preliminary study of a 30-state discretisation (using the middle-edge directions in the icosahedron) yields encouraging results (Rapaport 1985). Both the non-universal critical coupling and the magnetisation index of the three-dimensional system with nearest-neighbour coupling agree well with the corresponding data for the Heisenberg model.

The interpretation of this observation in renormalisation group language would be the statement that the anisotropy operator reducing the O(3) symmetry to a smaller discrete one is irrelevant in $d=3$. It is then natural to ask: are there simpler discretisations of the Heisenberg model still belonging to the same universality class in three dimensions? The answer to this question is the main content of the present comment.

In $d=2$ dimensions the picture changes dramatically. Investigation of the cubic anisotropy (Nienhuis *et al* 1983) has revealed that the corresponding anisotropy operator reducing the O(3) symmetry to that of a cube is relevant. This fact results in a non-trivial phase diagram, in contrast to the featureless phase structure of the O(3) model. The question of the evolution of the fixed point governing the ferromagnetic order-disorder transition as the number of allowed spin orientations increases was discussed by Margaritis *et al* (1986). Subsequently, the complete phase diagram of the icosahedral (12-state) and the dodecahedral (20-state) spin models was determined by Margaritis and Patkós (1986). In the closing part of our comment we extend this analysis to the 30-state ('middle-edge') discretisation proposed by Rapaport.

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The canonical (K, J) plane where we have studied the discrete models is given by the following Hamiltonian:

$$-\mathscr{H}/k_B T = K \sum_{x,e} (s_x s_{x+e} - 1) + J \sum_{x,e} ((s_x s_{x+e})^2 - 1). \tag{1}$$

The geometry of the discrete directions is described in previous publications (Rapaport 1985, Margaritis and Patkós 1986). For a qualitative comparison of the different models, the Migdal-Kadanoff real space renormalisation scheme has been applied to them (Migdal 1975, Kadanoff 1976). In the first step of our version the model with lattice spacing a is mapped non-linearly into a new effective model having b times larger cell size $(a \to ba)$. This mapping in terms of Boltzmann factors is given as

$$w'_{ij} = (w_{ij})^{b^{d-1}} \tag{2}$$

where i and j refer to the orientations of two neighbouring spins. The nearest-neighbour coupling in the new lattice results from a one-dimensional decimation giving for the final (as yet unnormalised) Boltzmann weights:

$$w''_{i_1 i_{b+1}} = w'_{i_1 i_2} w'_{i_2 i_3} \ldots w'_{i_b i_{b+1}}. \tag{3}$$

We impose the normalisation condition

$$w^{(\mathrm{ren})}_{i,i} \equiv 1$$

which gives

$$w^{(\mathrm{ren})}_{ij} = w''_{ij} / w''_{ii}. \tag{4}$$

This transformation has been performed for all the three models mentioned above in three dimensions with scale factor $b = 2$. The schematic phase diagrams have the same gross features as given in figure 1. The I_c-B boundary line is always Ising-type. Both the order-disorder and the partial order-disorder transitions (K_c-B and J_c-B

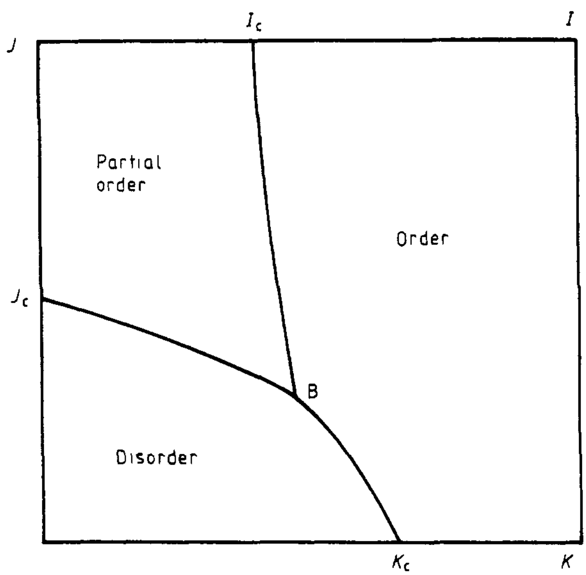

Figure 1. Common schematic phase diagram for the discrete spin models with 12, 20 and 30 states respectively.

lines respectively) are continuous. The largest eigenvalues of the renormalisation group transformations linearised around the corresponding fixed points show very little variation with the number of discrete directions, as can be seen from table 1. In particular, the y_T values obtained for the order-disorder transition from the different discrete spin models agree to better than 1% precision.

Without forgetting that the present approximation might distort the actual values of the indices, their close equality leads us to conjecture that the fixed point reached from the three models in a coupling space containing all kinds of anisotropy operators would be the same. Combined with the suggestion of Rapaport, this means that all these operators are irrelevant in $d = 3$.

This conjecture was tested by performing a Monte Carlo simulation of the icosahedral (12-state) discretisation on a 20^3 lattice. Although the cruder discretisation probably introduces stronger finite-size corrections, this size appeared to be sufficient to indicate that the above conjecture is indeed correct. The size of the lattice was essentially restricted by the small ESR-40 computer at our disposal at CRIP-Budapest.

In figure 2 the magnetisation ($M = 1/V\langle|\Sigma_i s_i|\rangle$) is shown as a function of K (J is set to 0). The data were obtained by averaging over 2500-3000 configurations/coupling in the critical region, the first 800 sweeps being used for equilibrating the system. The statistical errors are smaller than the size of the crosses. The data from the interval $K \in (0.7\text{-}0.8)$ were fitted to the scaling behaviour $M \sim (1 - K_c/K)^\beta$. A least-squares fit to the two parameters yields the following critical data:

$$K_c = 0.68 \pm 0.01 \qquad \beta = 0.36 \pm 0.05. \tag{5}$$

The results agree well with those from the 30-state discretisation (Rapaport 1985) and the critical characteristics of the original Heisenberg model (McKenzie *et al* 1982, Watson *et al* 1969). Larger lattices with a less rounded rise of the $M(K)$ function should reduce the errors in (5) (estimated by the range of parameters yielding good quality fits).

In two dimensions the increasing adherence of the approximation to the O(3) model is eventually signalled by the decreasing tendency of the y_T eigenvalue as the density of the allowed directions increases (Margaritis and Patkós 1986):

$$\nu_{30} > \nu_{20} > \nu_{12}.$$

The entries in table 1 for $d = 2$ fully confirm this expectation, both for the order-disorder

Table 1. The leading thermal exponents of the discrete spin models in the (K, J) plane and in $d = 2, 3$ dimensions as determined from a Migdal-Kadanoff iteration.

	y_T (order-disorder)		
d	Icosahedron	Dodecahedron	30-state model
2	1.561	1.263	1.186
3	1.814	1.819	1.812
	y_T (partial order-disorder)		
2	1.962	1.702	1.336
3	2.278	2.139	2.057

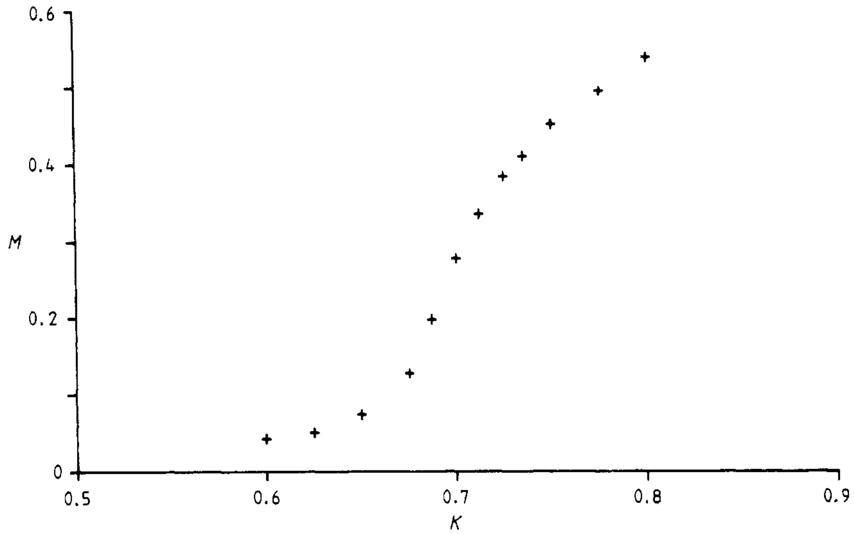

Figure 2. Monte Carlo results for the magnetisation in the icosahedral spin model on a 20^3 lattice.

and the partial order-disorder transitions. We have also noticed the monotonic tendency pushing the critical K_c and J_c values higher (and the critical temperatures towards zero). It is obvious that the different discretisations belong to different universality classes in $d = 2$.

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