Estimation of the order-parameter exponent of critical cellular automata using the enhanced coherent anomaly method

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The stochastic cellular automaton of rule 18 defined by S. Wolfram [Rev. Mod. Phys. **55**, 601 (1983)] has been investigated by the enhanced coherent anomaly method. A reliable estimate was found for the β critical exponent, based on moderate sized ($n \leq 7$) clusters.

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Calculating critical exponents of second order phase transitions is a challenging problem. For nonequilibrium systems, generalization of equilibrium statistical physics methods is under developement. Among the most notable analytical tools are series expansion [1], transfer matrix diagonalization [2], and the mean-field renormalization-group method [3].

In a series of earlier papers [4–6], we have shown how the generalization of the mean-field technique with appropriate extrapolation can be used to describe the critical properties of cellular automata (CA) phase transitions.

The generalized mean-field approximation (GMF) first proposed for dynamical systems by Gutowitcz *et al.* [7] and Dickman [8] is shown to converge slowly at criticality. In this method we set up equations for the steady state of the system based on n-point block probabilities. Correlations with a range greater than n are neglected. By increasing n from 1 (traditional mean field) step by step we take into account more and more correlations and get better approximations. The GMF approximation can be used as a basis of a coherent anomaly method (CAM) calculation, and it gives a reasonably good β exponent for a dynamical system with large n (>10) [9]. In this Brief Report I show how an improved version of the CAM proposed very recently [10] works on cellular automata.

The essence of the CAM [11] is that the solution for singular quantities at a given (n) level of approximation [$Q_n(p)$] in the vicinity of the critical point is the product of the classical singular behavior multiplied by an anomaly factor [$a(n)$], which becomes anomalously large as $n \to \infty$ (and $p_c^n \to p_c$):

$$Q_n \sim a(n)(p/p_c^n - 1)^{\omega_{\rm cl}}, \qquad (1)$$

where p is the control parameter and $\omega_{\rm cl}$ is the classical critical index. The divergence of this anomaly factor scales as

$$a(n) \sim (p_c^n - p_c)^{\omega-\omega_{\rm cl}}, \qquad (2)$$

thereby permitting the estimation of the true critical exponent ω, given a set of GMF approximation solutions. However, such an estimation depends to some extent on the choice of the independent parameter ($p \leftrightarrow 1/p$). To avoid this a corrected CAM was proposed [10], based on a new parameter,

$$\delta_n = (p_c/p_c^n)^{1/2} - (p_c^n/p_c)^{1/2}, \qquad (3)$$

such that Eq. (3) is invariant under $p \leftrightarrow p^{-1}$. This parametrization gives better estimates for the critical exponents of the three-dimensional Ising model [10].

My target system for this kind of calculation was the one-dimensional, stochastic rule 18 CA [12]. This range-1 cellular automaton rule generates a 1 at time t only when the right or the left neighbor was 1 at $t-1$:

$t-1$:	100	001
t:	1	1

with probability p. In any other case the cell becomes 0 at time t. The order parameter is the concentration (c) of 1s. For $p < p_c$ the system evolves to an absorbing state ($c = 0$). For $p \geq p_c$ a finite concentration steady state appears with a continuous phase transition. This transition is known to belong to the universality class of directed percolation (DP) [13]. At $t \to \infty$ the steady state can be built up from 00 and 01 blocks [14]. This permits one to map it onto stochastic rule 6/16 CA with the new variables $01 \to 1$ and $00 \to 0$:

$t-1$:	0 0	0 1	1 0	1 1
t:	0	1	1	0

and the GMF equations can be set up by means of pair variables. In an earlier work [4] this was performed up to the order $n = 6$, and Padé extrapolation was applied to the results. Our best estimate for critical data was $p_c = 0.7986$ and $\beta = 0.29$.

Now, I have extended the GMF calculations up to $n = 7$ (see Table I) with the help of the symbolic

TABLE I. GMF calculation results for pair approximation data.

n	p_c^n	$a(n)$
1	0.5000	0.5000
2	0.6666	1.5000
3	0.7094	2.3484
4	0.7413	2.8816
5	0.7543	3.5345
6	0.7656	4.2545
7	0.7729	4.8463

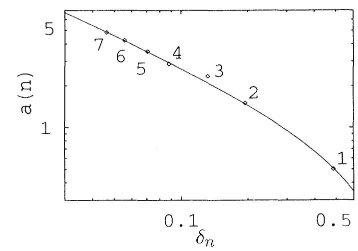

FIG. 1. Result obtained by applying the improved CAM method on n-pair ($n = 1, \ldots, 7$) approximation data. The logarithm of the anomaly coefficient $a(n)$ is plotted versus the logarithm of the improved independent variable δ_n. Fitting was done according to Eq. (4).

MATHEMATICA software. This required the setting up and solution of a set of nonlinear equations of 72 variables. I obtained $p_c^7 = 0.7729$, which is still 5% off the result obtained by steady state simulation, $p_c = 0.8086(2)$ [15] or from the more accurate time dependent simulation data, $p_c = 0.8094(2)$ [16].

The CAM analysis of $[a(n), \delta_n]$ data was done, taking into account the correction term

$$a(n) = b\ \delta_n^{\beta-\beta_{\rm cl}} + c\ \delta_n^{\beta-\beta_{\rm cl}+1} \tag{4}$$

and examining the stability of the solution by omitting different points from the ($n = 1, \ldots, 7$) data set. For the fitting $\beta_{\rm cl} = 1$ and $p_c = 0.8094$ were used. As was pointed out in Ref. [10] the CAM data may contain departures from ideal scaling; moreover, there is no clear dependence on the order of the approximations. I found relatively stable estimates using the correction formula (4) on the data set with the omission of the $n = 3$ point. The $n = 3$ approximation result does not fit into the $\ln(\delta_n)$-$\ln[a(n)]$ curve either (see Fig. 1). Table II shows the stability of the results, with the mean $\beta = 0.2796(2)$ calculated from them. This compares very well with the value $\beta = 0.2769(2)$ obtained by Dickman and Jensen [1] from series expansion. For the CAM calculation based on p or $1/p$ independent variables the results differ by ± 0.005 from the present enhanced version β estimates.

TABLE II. CAM calculation results for pair approximation data.

Data	β
1-2-4-5	0.273
1-2-4-5-6	0.271
1-2-4-5-6-7	0.282
1-2-4-5-7	0.285
1-2-4-6-7	0.275
1-2-5-6-7	0.310
1-4-5-6-7	0.275
2-4-5-6-7	0.266
Mean	0.2796(2)
Padé extrapolation, Ref. [4]	0.29
Simulation, Ref. [15]	0.285(5)
Series expansion for DP, Ref. [1]	0.2769(2)

Another critical model with non-DP universality, the nonequilibrium kinetic Ising model, has been examined with the enhanced CAM method, and the β exponent estimate is in agreement with the simulation results [17].

The conclusion of this study is that the enhanced version CAM method with careful data analysis gives good estimates for the critical exponent for moderate $n < 10$ level GMF approximations. Calculation of the $n = 5, 6, 7, \ldots$ GMF approximations is possible on moderate sized workstations. The solution of the $n = 7$ level approximation took about 10 h CPU time on a SUN Sparc-10 computer. This provides an efficient analytical tool for exploring universalities of nonequilibrium systems.

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