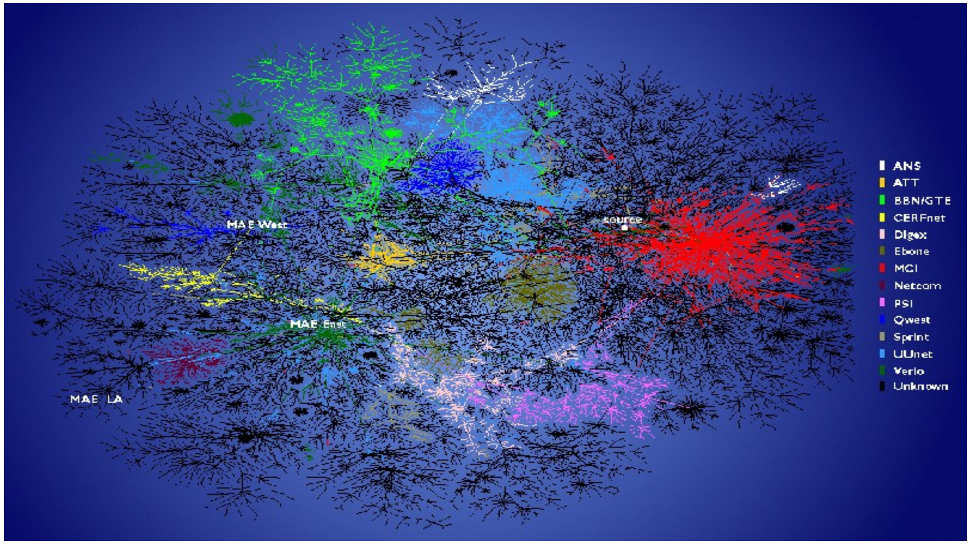
Griffiths phases of contact processes in complex networks

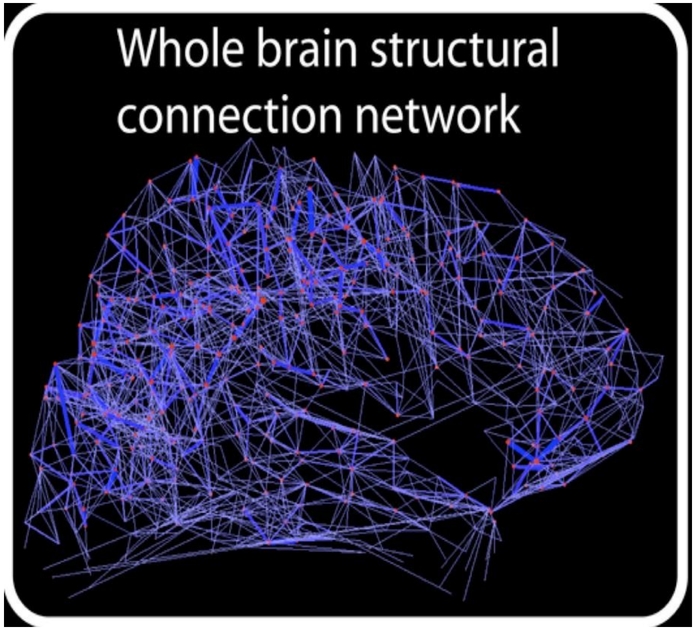
Géza Ódor

RESEARCH INSTITUTE FOR TECHNICAL PHYSICS AND MATERIALS SCIENCE (MFA) BUDAPEST

- Exploration of complex networks is flourishing since ~2000 (Barabási & Albert)
- Dynamical systems living on networks is of current interest
- Research of disordered materials is intensifying
- Open: Complex networks + quenched disorder ?
- Origin of non-universal (dynamic) scaling behaviors ?

Internet

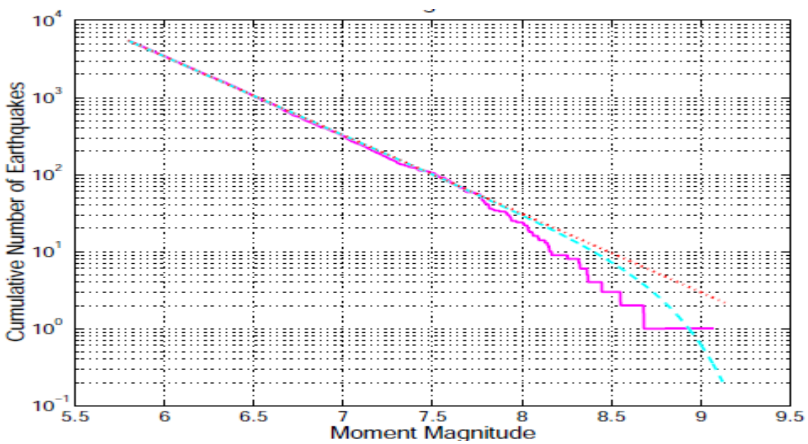


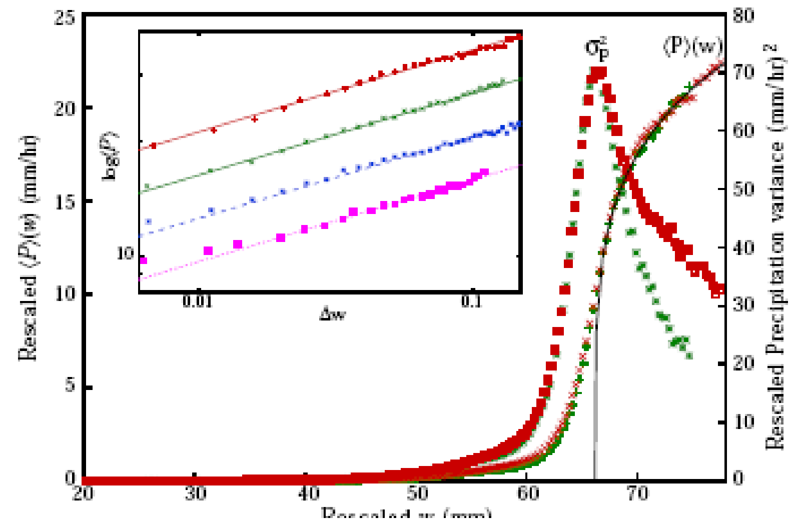


Diffusion spectrum imaging

Power-law scaling in nature

• Earthquake size distribution



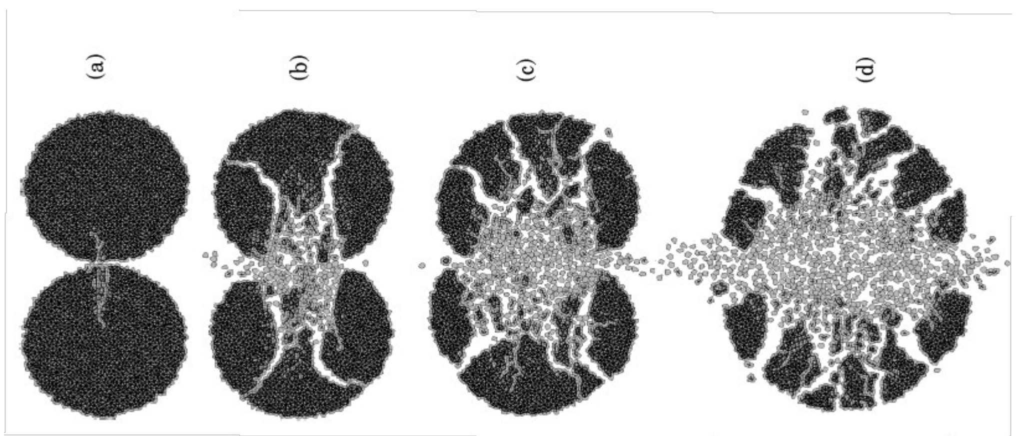


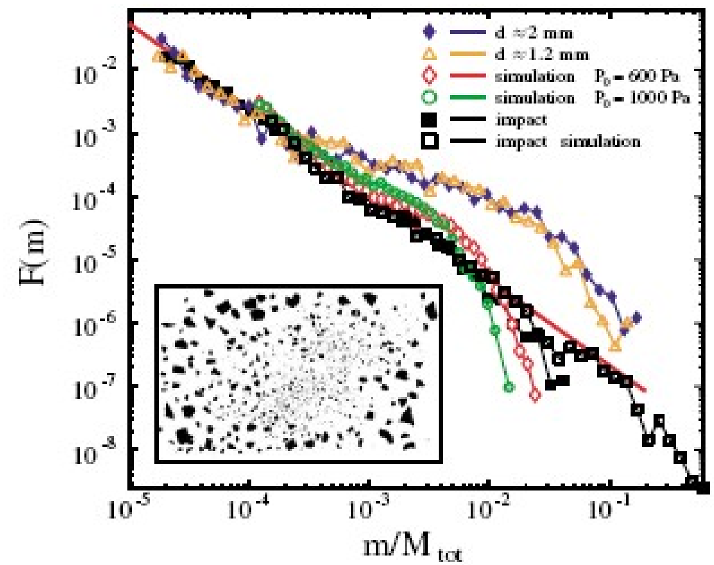
Meteorology and Climatology:

O.Petres and D. Neelin, Nature Phys. 2 (2006) 393

Damage formation by collisions, explosions

(F. Kun & H. Hermann)





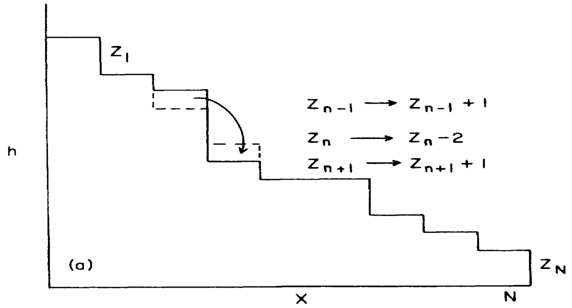
•And many more...

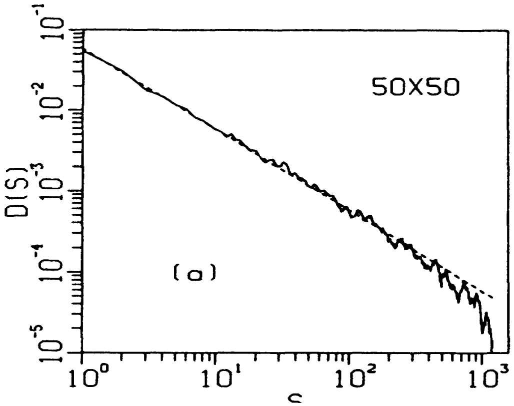
What is the origin ?

“Self-Organized Criticality” (SOC) ?

Bak-Tang-Wiesenfeld sand pile model (1987) CA :

Add a grain of sand:
$h(x,y) \rightarrow h(x,y) + 1$
And avalanche if: $h(x,y) > h_c$:
$h(x, y) \rightarrow h(x, y) - 4$
$h(x \pm 1, y) \rightarrow h(x \pm 1, y) + 1$
$h(x, y \pm 1) \rightarrow h(x, y \pm 1) + 1$

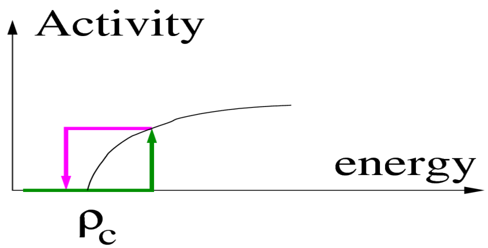




The term SOC usually refers to a mechanism of *slow energy accumulation* and *fast energy redistribution*, driving a system toward a critical state.

Prototype: sand pile model

Self-tuning to critical point:

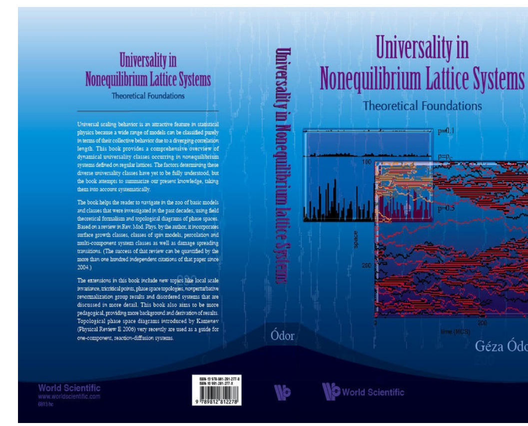


Stochastic sand pile models can be mapped onto ordinary nonequilibrium criticality at **phase transition to absorbing phases** (*Dickman et al. 2000*)

Scaling-universality classes of basic models

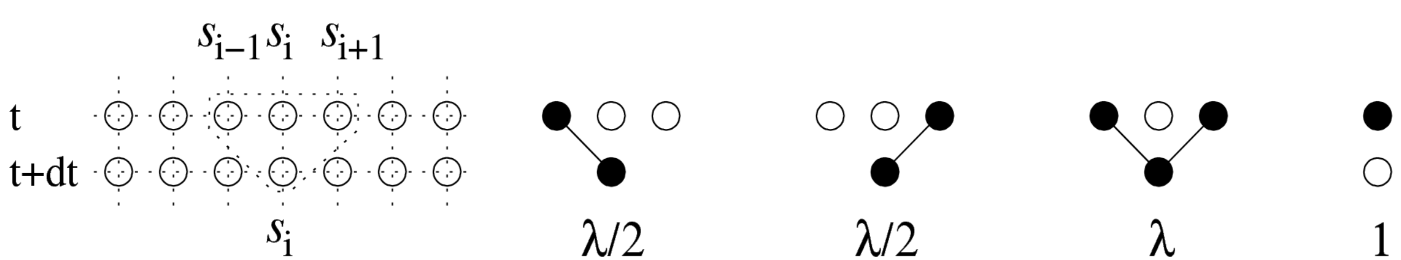
Universality classes in complex system due to : $\xi \to \infty$
i.e: near critical points or in system with currents (nonequilibrium)

Basic models can be classified by universal scaling behavior according to some global conditions: dimensions, symmetries,..., topological constraints :
G. Ódor: Universality in nonequilibrium system (World Scientific 2008), Rev. Mod. Phys. 2004



What prevents the observation of universal behavior ?

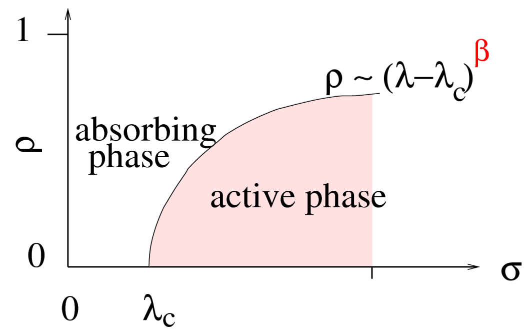
Prototype model describing epidemic or activity spreading : **C**ontact **P**rocess (**CP**) (**1d**):



Contact process, scaling, DP

- **Order parameter**: density of active sites: $\rho(t) = \frac{1}{L^d}\sum_{\mathbf{r}} \langle n_{\mathbf{r}}(t)\rangle \qquad \rho_{\rm stat} = \lim_{t\to\infty}\rho(t)$

- **Critical phase transition**
  between active $\leftrightarrow$ absorbing states:

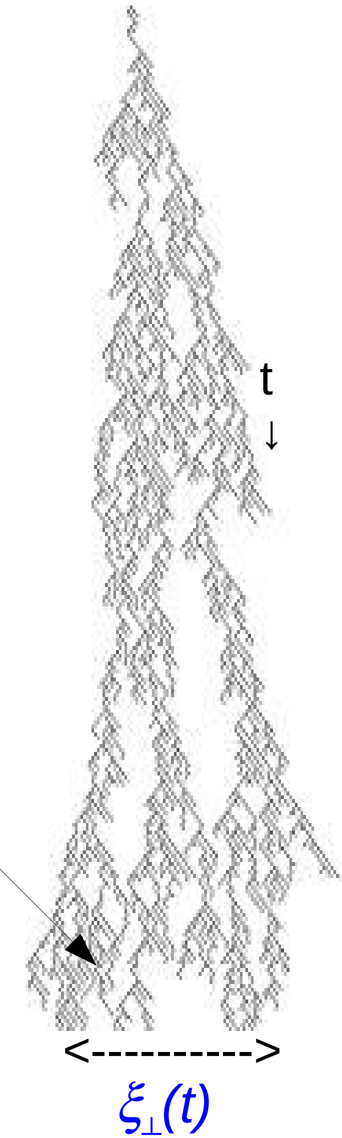


- **Scaling behavior**

Correlation length diverges anisotropically : $\xi_\perp \sim |\Delta|^{-\nu_\perp}$, $\xi_\parallel \propto |\Delta|^{-\nu_\parallel}$

Independent critical exponents: β , $\nu_\perp$, $\nu_\parallel$

Density decay: $\rho(t) \sim t^{-\alpha}$ *where:* $\alpha = \beta / \nu_\parallel$

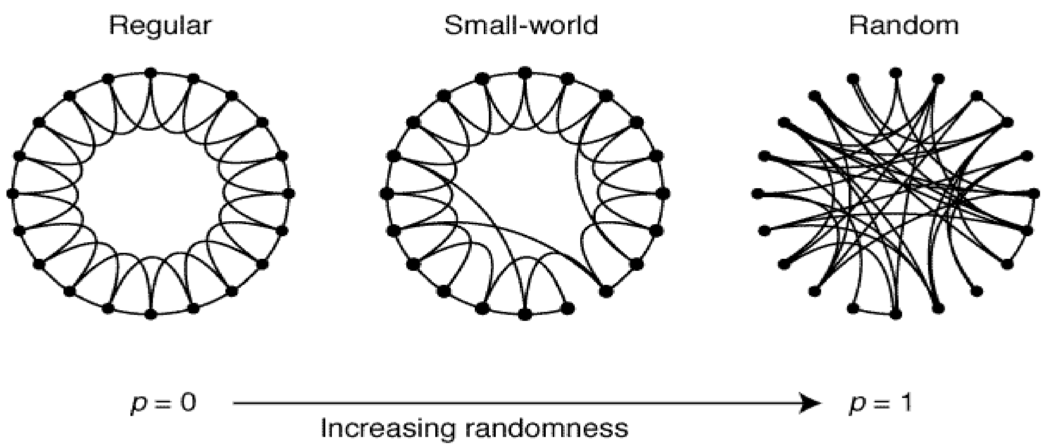


- **D**irected **P**ercolation, **DP** Universality Class

Robust class of epidemic, information, … etc. spreading
well known in *d = 1,2,3,4* ... Eucledean dimensions

Basic network models

From regular to random networks:



Erdős-Rényi ($p = 1$)

Degree (k) distribution :

$N \rightarrow \infty$ node limit:

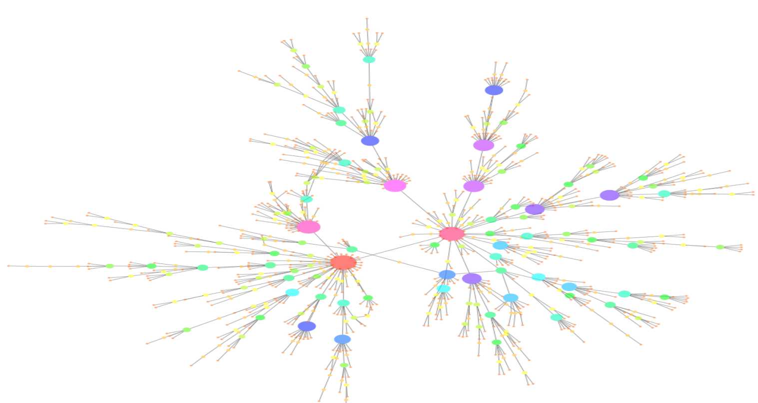
$$P(k) = e^{-<k>} <k>^{k} / k!$$

Topological dimension: $N(r) \sim r^{d}$

Above perc. thresh.: $d = \infty$

Below: $d = \mathbf{0}$

Scale free networks:



Barabási-Albert

Degree distribution:

$$P(k) = k^{-\gamma} \quad (2 < \gamma < 3)$$

Topological dimension: $d = \infty$

Focus on dynamical systems living on networks: Fast dynamics is expected

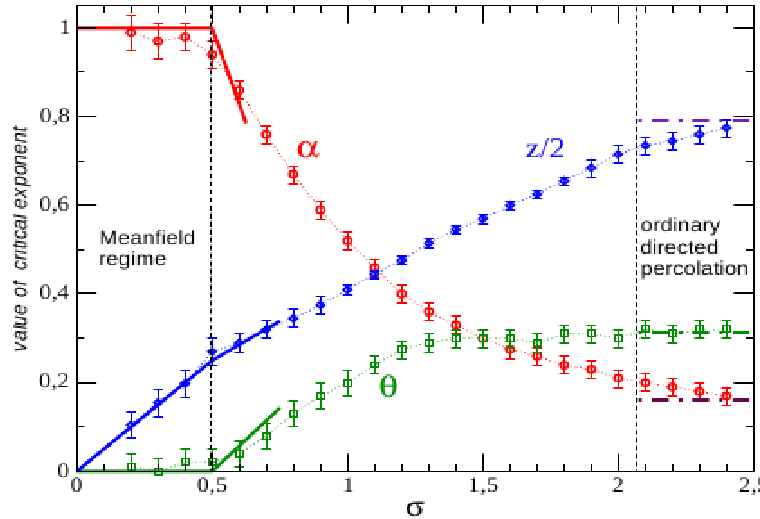
How do networks differ from Eucledean lattices in general ?

1) **Small diameter** $\rightarrow$ *long-range interactions* $\rightarrow$ *weak fluctuations* $\rightarrow$ *mean-field behavior*

DP + *Lévy flight :* $P(r) \sim r^{-(d+\sigma)}$

Continuously changing exponents :

DP $\rightarrow$ mean-field-**DP** *as:* $\sigma \rightarrow 0$



(Janssen et al, Hinrichsen et al 1999)

2) **Heterogenicity** (i.e. aperiodicity, changing degree ***k*** …) $\rightarrow$ Other critical points ...

3) **Disorder (quenched)** $\rightarrow$ Other critical points ...
Griffiths phases, activated scaling ?

Up to now mainly ***1)*** + ***2)*** cases have been investigated by dynamical models

The heterogeneous mean-field approximation omits quenched disorder, describes annealed disorder, what we have learnt :

Topology is relevant :

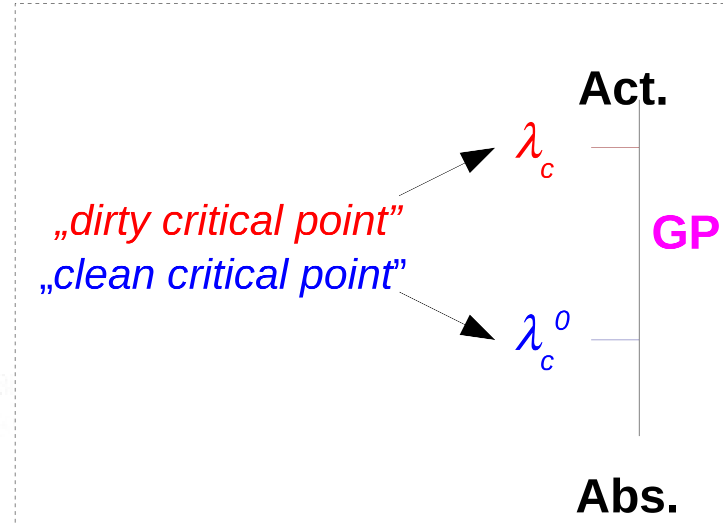
The effect of topological disorder on scale-free: $P(k) \sim k^{-\gamma}$ networks:

$\lambda_c = \langle k \rangle / \langle k^2 \rangle = 0$ if $2 < \gamma < 3$ (Ising, CP …) *Always active Contact Process;*

$\lambda_c > 0$ *if* $\gamma = 3$*, or finite size, or weakened (weighted) links:* γ *dependent exponents*

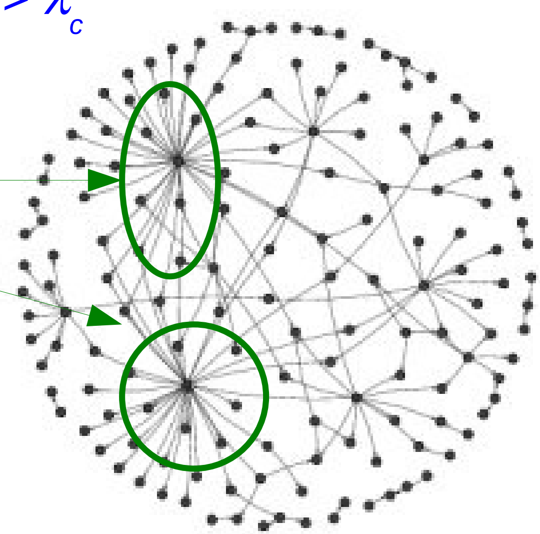
Rare Region argument for Q-disordered CP

- Fixed (quenched) disorder/impurity **_changes_ the local birth rate** $\Rightarrow \lambda_c > \lambda_c^0$



- **Locally active**, arbitrarily large Rare Regions

  in the inactive phase due to the *inhomogeneities*



- Probability or RR of size L_R:

  $w(L_R) \sim \exp(-c\, L_R^d)$

  Contribute to the density: $\rho(t) \sim \int dL_R \; L_R^d \; w(L_R) \; \exp[-t / \tau(L_R^d)]$

- **For** $\lambda < \lambda_c^0$: conventional (exponentially fast) decay

- **At** λ_c^0 the characteristic time scales as: $\tau(L_R) \sim L_R^{-Z}$ $\Rightarrow$ saddle point analysis:

  $$\ln \rho(t) \sim t^{d/(d+Z)}$$ *(stretched exponential)*

- **For** $\lambda_c^0 < \lambda < \lambda_c$: $\tau(L_R) \sim \exp(b\, L_R)$: **Griffiths Phase**

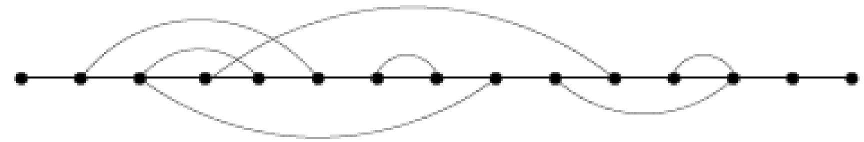
  $\Rightarrow$ saddle point analysis: $\rho(t) \sim t^{-c/b}$ *(continuously changing exponents)*

- **At** λ_c Ultra slow time dependences : $\rho(t) \sim \ln(t)^{-\alpha}$

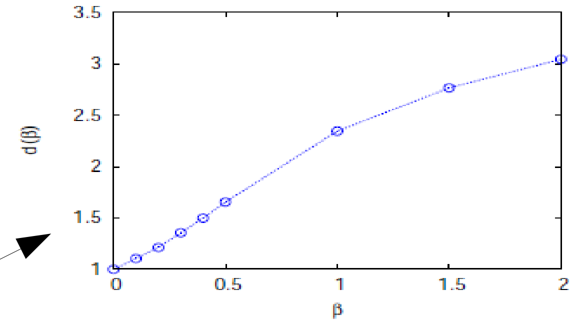
CP + Topological disorder results

Generalized Small World networks: $P(l) \sim \beta\, l^{-2}$

(link length probability)



- *Top. dim:* $N(l) \sim l^{d}$ $d(\beta)$ **finite:**



$\lambda_c(\beta)$ *decreases* monotonically from

$\lambda_c(0) = 3.29785$ *(1d* **CP***) to:*

$\lim_{\beta \to \infty} \lambda_c(\beta) = 1$ towards mean-field **CP** value

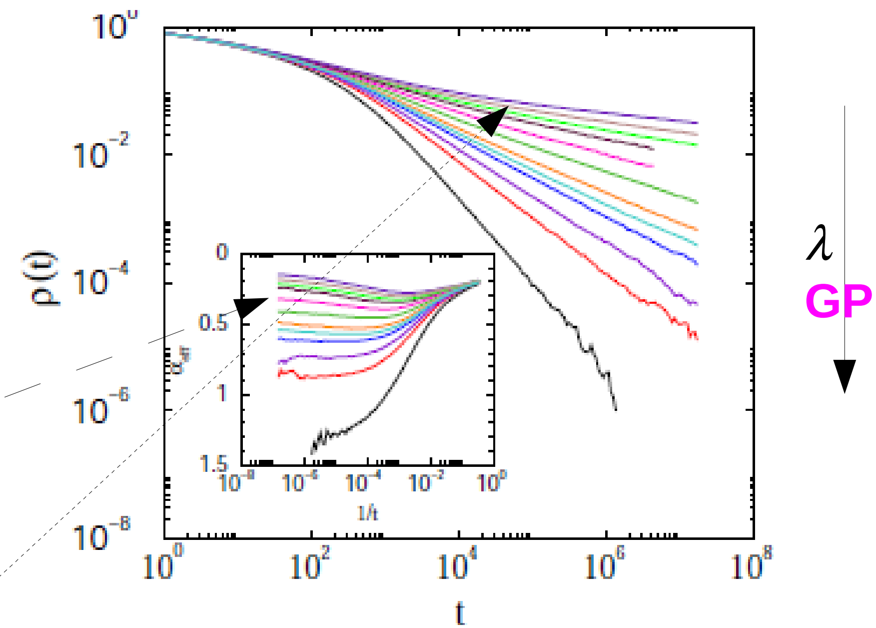
$\lambda < \lambda_c(\beta)$ inactive, there can be

locally ordered, **rare regions** due to more than avarage, active, incoming links

- **Griffiths phase:** *λ – dep. continuously changing dynamical power laws:*

for example : $\rho(t) \sim t^{-\alpha(\lambda)}$

Logarithmic corrections !



- **Ultra-slow** ("activated") scaling: $\rho \propto \ln(t)^{-\alpha}$ at λ_c

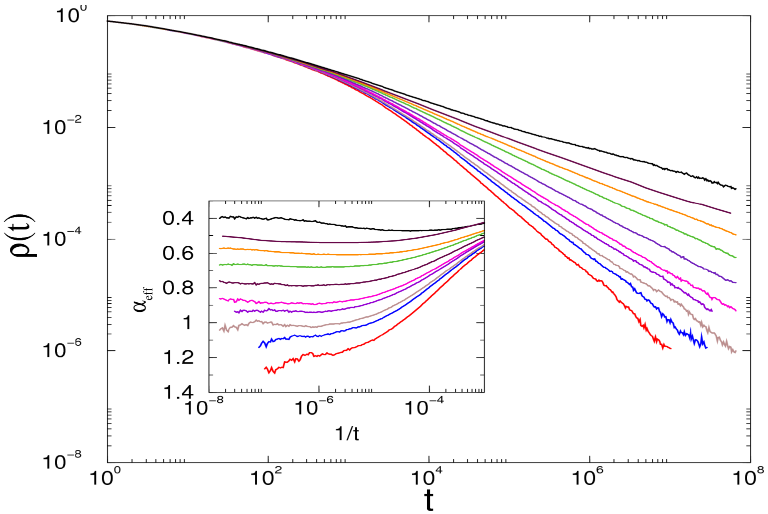
FIG. 3: Density decay in Benjamini-Berger networks with $s = 2$ and $\beta = 0.2$ for different values of λ (from top to bottom: $2.81, 2.795, 2.782, 2.77, 2.75, 2.73, 2.71, 2.70, 2.69, 2.67, 2.65, 2.6$). Straight lines lie in the Griffiths phase. Inset: Corresponding effective exponents, illustrating the presence of corrections to scaling.

- As $\beta \to 1$ Griffiths phase shrinks/disappears

- **Same results for cubic regular random networks %**

Generelaized SW networks with GP

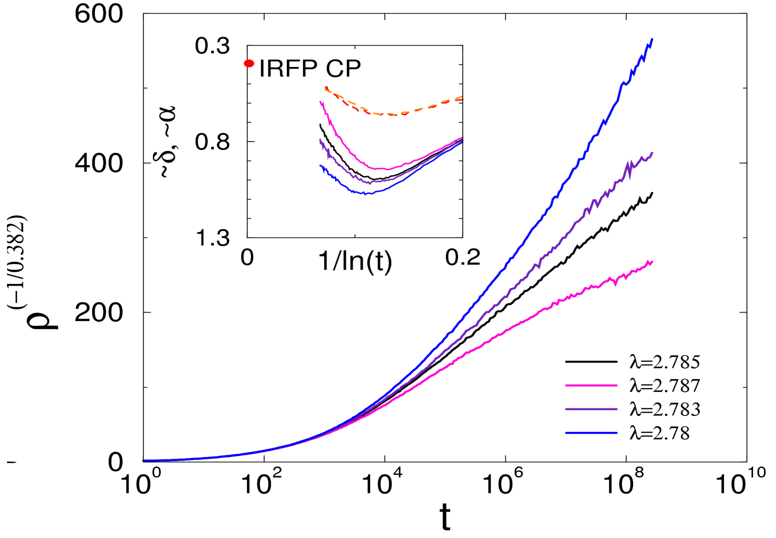
- **CP on cubic regular random networks**
  L* = *10 ^{7} sized simulations:
  Griffiths Phase (strong log. corrections)



- **For *general:* *P(l)* ~ β *l*** $^{-s}$

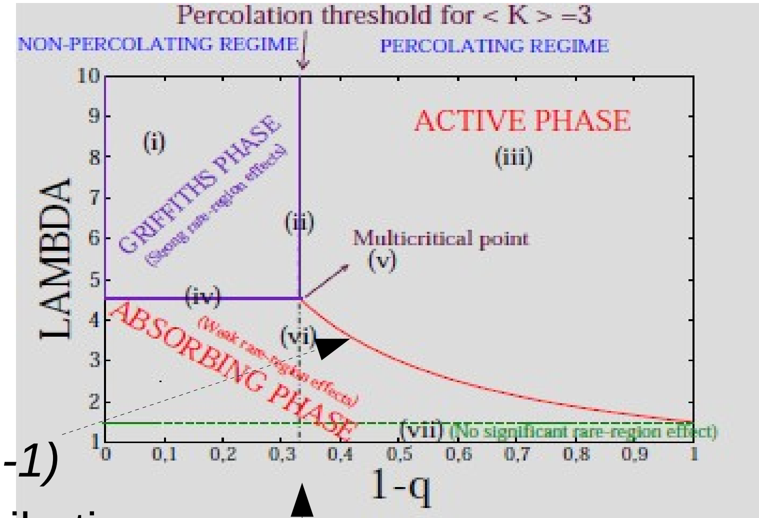
s* = *0 **mean-field behavior**
s* < *2 ***d*** **= ∞,** power-laws (Lévy) with no GP
s* > *2 mean edge length is finite,
d* = *1 CP with topological disorder
rare regions: subgraphs with more internal edges than the average
Griffiths Phase
Conjecture: IRPF of the disordered CP

$$\rho(t) \sim [\ln t]^{-\overline{\delta}} \qquad \overline{\delta} = 0.38197\ldots$$
$$[\rho(t)]^{-1/\overline{\delta}} \sim \ln t$$
$$\ln \rho(t) \sim -\overline{\delta} \ln(\ln t)$$
$$\overline{\delta}_{\rm eff}(t) = -\frac{d \ln \rho}{d \ln(\ln t)}$$



For other networks ?

Above percolation threshold
rare regions do not emerge →
Do not exepect Griffiths phase in nets
like Barabási-Albert vagy Erdős-Rényi
if $<k>$ is large



$r = 0$

Q-CP on Erdős-Rényi network $<k> = 3$

Mean-field : $\lambda_c = 1$, Pair mean-field: $\lambda_c^2 = <k> / (<k> -1)$

+ reduced reaction rates *with* binary probability distribution:

$P[\lambda(x)] = \mathbf{q}\ \delta[\lambda(x) - \mathbf{r}\ \lambda] + (\mathbf{1} - \mathbf{q})\ \delta[\lambda(x) - \lambda]$ $\quad q_{perc} = 2/3$

Rare Region theory $\quad \lambda_c^2 = <k> / (<k> -1)\ [1 - q(1-r)]$

(i) **GP.** Cluster size distribution : $P(s) \sim s^{-3/2} exp[-s(p-1-ln(p))]$, *where:* $p = <k>(1-\mathbf{q})$

$\rho(t) \sim \int ds\ s\ P(s)\ exp[-t / \tau(s)]$ $\quad$ *where:* $\tau(s) \cong t_o\ exp[A(\lambda)s]$

r = 0 case:

Saddle-point approximation: $\rho(t) \sim t^{-\gamma}$ $\gamma = (p-1-ln(p))\ A(\lambda)$ *cont. changeing dyn. exps.*

(ii) $q = q_{perc}$: $\rho(t) \sim [ln(t/t_0)]^{-1/2}$ *ha* $\lambda > \lambda_c$ *activated scaling*

(iii) Active phase: Giant component, MF+Griffiths eff., If $q < q_{perc}$ critical line: $\rho(t) \sim 1/t$

(iv) $q > q_{perc}$ $\lambda = \lambda_c$ ∃ **RR** almost critical, streched exponential decay

(v) Multi-critical point: power-law decay

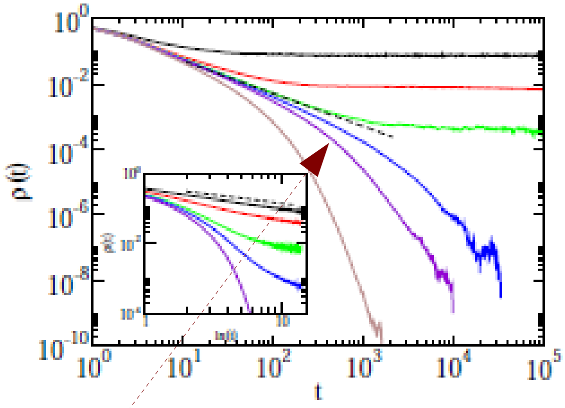
(vi) Abs phase: $\lambda_c(q_{perc}) > \lambda > \lambda_c(q=0)$ ∃ supercritical **RR** *if:* $q_{loc} < q_{perc}$, but weak effects

(vii) $\lambda < 1.5$: **RR free region** : exponential decay

r > 0 : Similar phase-diagram, but active phase goes up to $q=1$

Simulation results for Q-CP on Erdős-Rényi networks

N $\leq 10^7$ *<k> = 3* Decay from fully active state,
Cluster growth from a single seed
Same asymptotic behavior : *Rapidity* symmetry

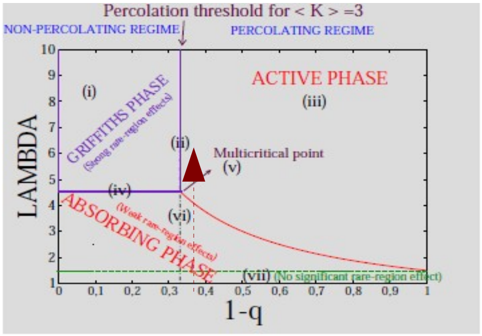


Clear case: $\lambda_c\,(q=0) \approx 1.5$, $\lambda_c\,(q=1) \approx 30$ agrees with:

$\lambda_c^{\,2} = \langle k\rangle / (\langle k\rangle - 1)\,[1 - q(1-r)]$ pair approximation,

$\rho(t) \sim 1/t$

0 < q < 1: Agreement with the theory:



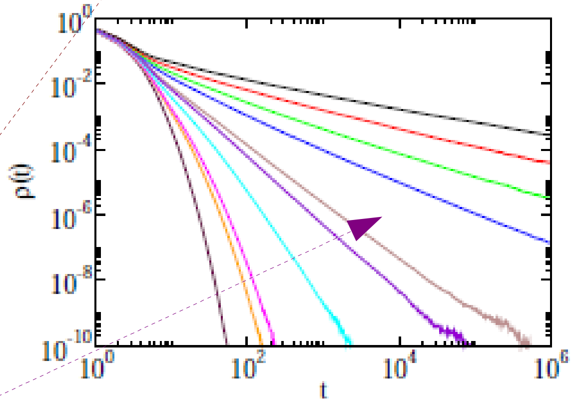


FIG. 1: Average activity density $\rho(t)$ vs time t for ER networks with $\langle k\rangle = 3$, $r = 0$, and $N = 10^5$. λs are ordered from top to bottom in all panels. **(a) Upper panel**: $q = 0.6$, and $\lambda = 5, 3.8, 3.6, 3.55.3.5, 3.3$. The dashed line is proportional to t^{-1}. **(a) Inset**: ρ vs $\ln(t)$ for $q = 2/3$; $\lambda = 10, 7, 5, 4.5, 4$; the dashed line is proportional to $\ln(t)^{-1/2}$. **(b) Lower panel**: $q = 0.9$, and $\lambda = 50, 30, 20, 15, 10, 9, 7, 5, 4.5, 2.7$. Straight lines lie in the Griffiths phase.

***r* = 0,** *q < 2/3 :* Mean-field DP critical transition

q > 2/3 : No active phase:

$\lambda > \lambda_c(q = q_{perc} = 2/3) = 4.5$: $\rho(t) \sim t^{-\gamma}$ **GP**

***r* > 0** *:* similar phase diagram,
active phase goes up to *q =1*

Summary

- Quenched **disorder** increases the occurrences of **non-universal scaling behavior**:   Extended Griffiths Phases must occur in nature

- **Slow** (algebraic, logarithmic) dependence is expected in many dynamical systems defined on networks ($\leftrightarrow$ **Fast** dynamics in **pure** network models)

  Understanding algebraic slow forgetting times of working memory in a **simple model of brain** (see the paper/talk by Johnson et al)

  In other epidemic/information spreading models living on heterogeneous random networks with **finite topological dimensions**,

- New Hungarian-Spanish-Italian collaboration has been set up with the first results:

  Griffiths phases on complex networks: ***M. A. Munoz, R. Juhász, C. Castellano*** *and G. Ódor arXiv:1009.0395, PRL in press.*

- Acknowledgments: **HPC2 Europe, OTKA, OSIRIS FP7**