Griffiths phases, localization and burstyness in network models

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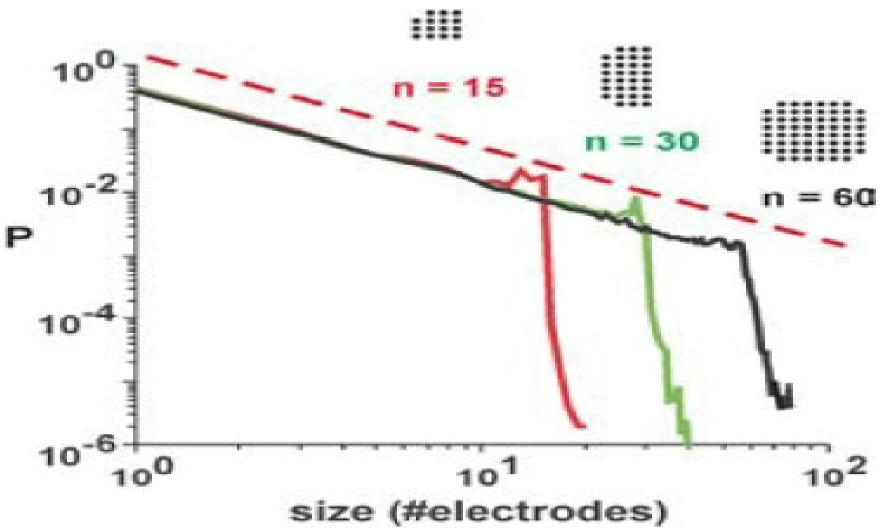
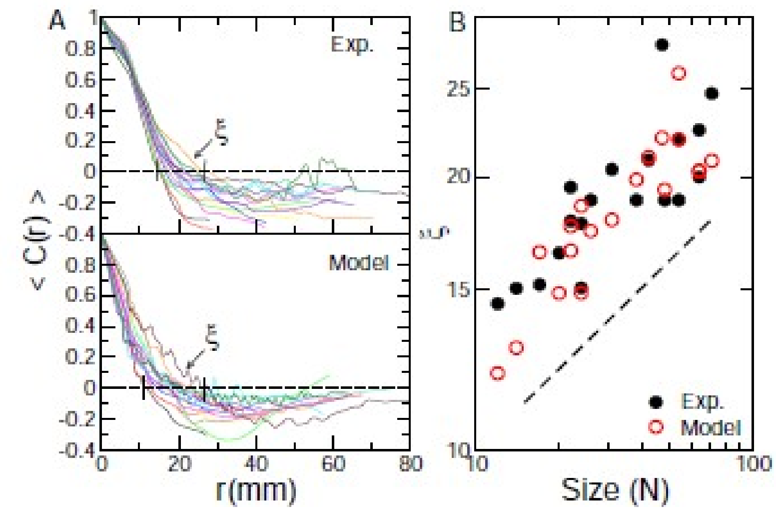
Power-laws & critical, slow dynamics observed in networks

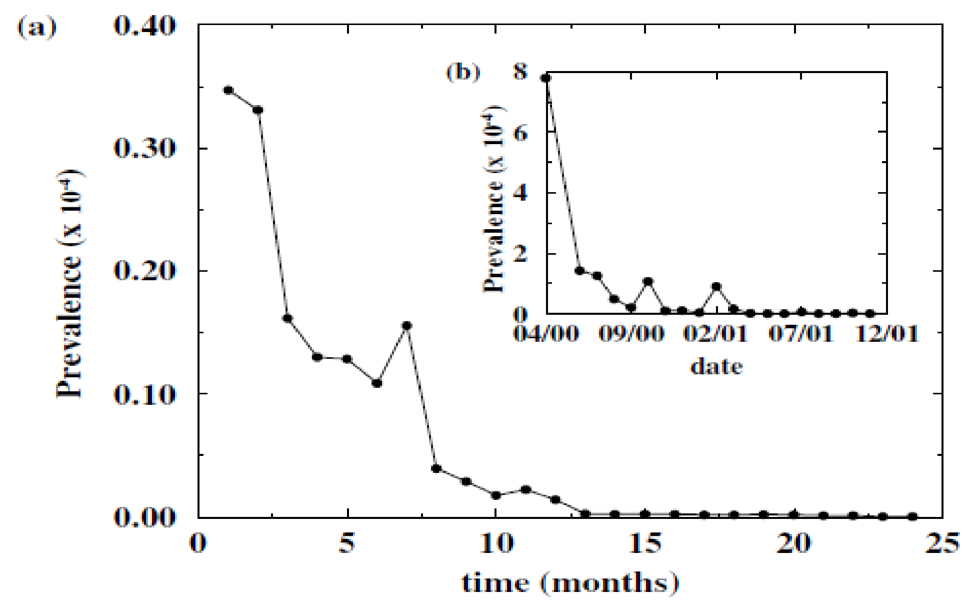
- **Brain** : PL size distribution of neural avalanches
  G. Werner : Biosystems, 90 (2007) 496,



- *Correlation length (ξ) diverges*
  Haimovici et al PRL (2013) :
  *Brain complexity born out of **criticality**.*



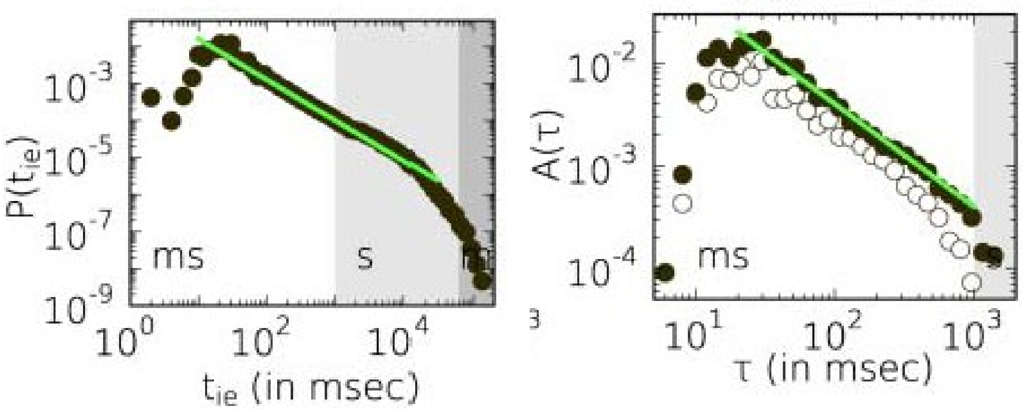
- Internet: worm recovery time is **slow**:



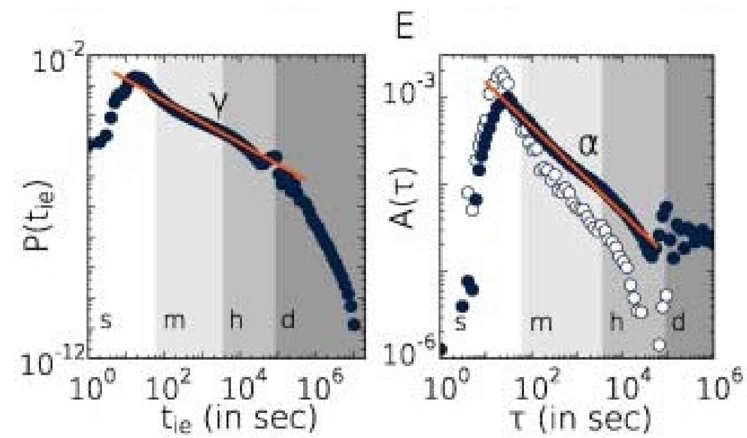
How can we explain power-law dynamics in network models ?

Burstyness observed in nature

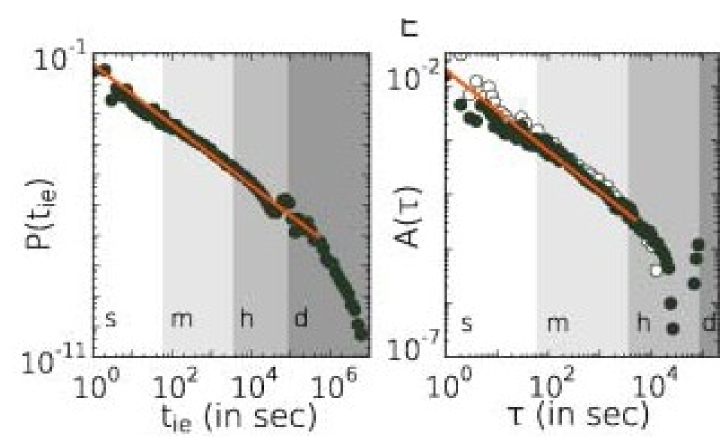
- **Brain** : PL inter-event time distribution of neuron firing sequences & Autocorrelations
  *Y. Ikegaya et al.: Science, **304** (2004) 559,*
  *N. Takahashi et al.: Neurosci. Res. **58** (2007) 219*



- ***Mobile call:*** *Inter-event times and Autocorrelations*
  *Karsai et al. PRE **83** (2013) 025102 : Small but slow world ...*



- **Internet**: Email sequences:
- And many more ….
- **Models** exist to explain internal non-Markovian behavior of agents *(Karsai et al.: Sci. Rep. **2** (2012) 397)*



*J. Eckmann et al.: PNAS **101** (2004) 14333*

- Can this occur by **the collective behavior** of Markovian agents **?**

Scaling in nonequilibrium system

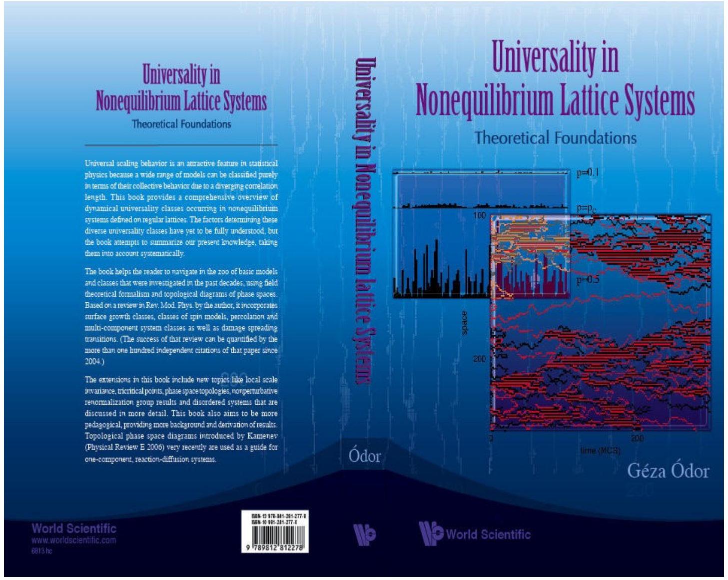
Scaling and universality classes appear in complex system due to : $\xi \to \infty$
i.e: near critical points, due to currents, ...

Basic models are classified by **universal scaling behavior** in Euclidean, regular system

- **Why don't we find these critical universality classes in dynamical network models ?**

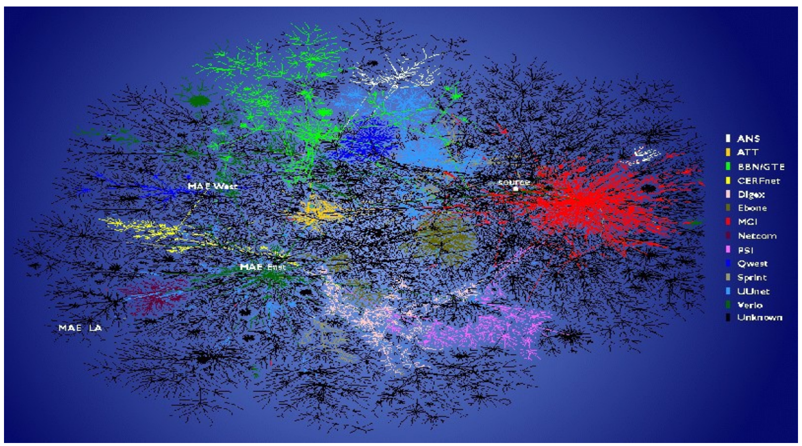
- **Power laws are frequent in nature**
  ↔ Tuning to critical point (SOC) ?

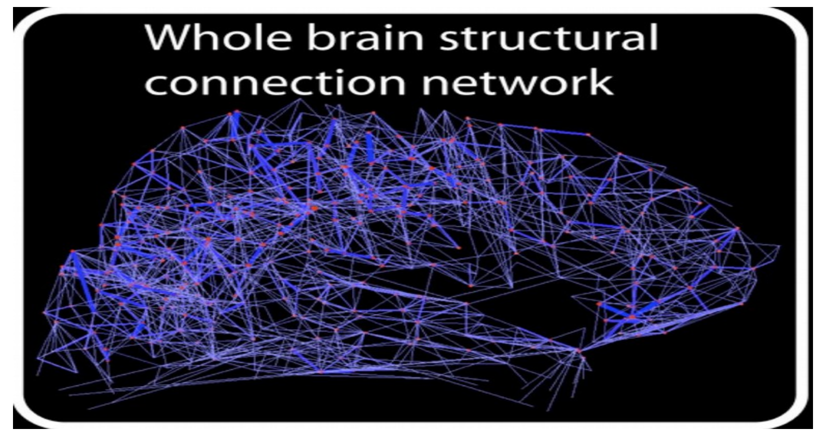
I'll show a possible way to understand these in case of quasi-static networks



Dynamical models on networks

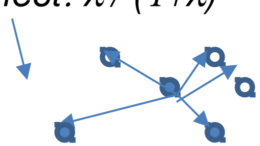
Small world networks:
Expectation: mean-field type behavior with fast dynamics



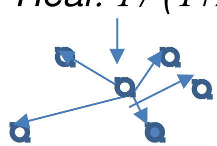


*Prototype: **S**usceptible-**I**nfected-**S**usceptible (**SIS**) two-state model:*

Infect: $\lambda / (1+\lambda)$

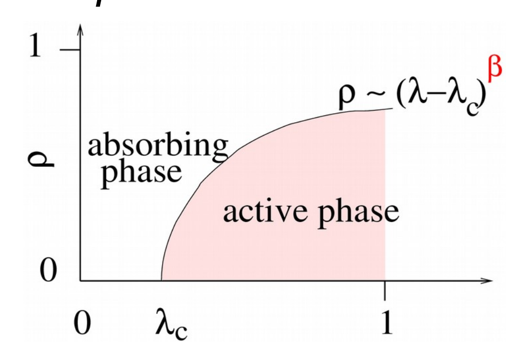


Heal: $1 / (1+\lambda)$



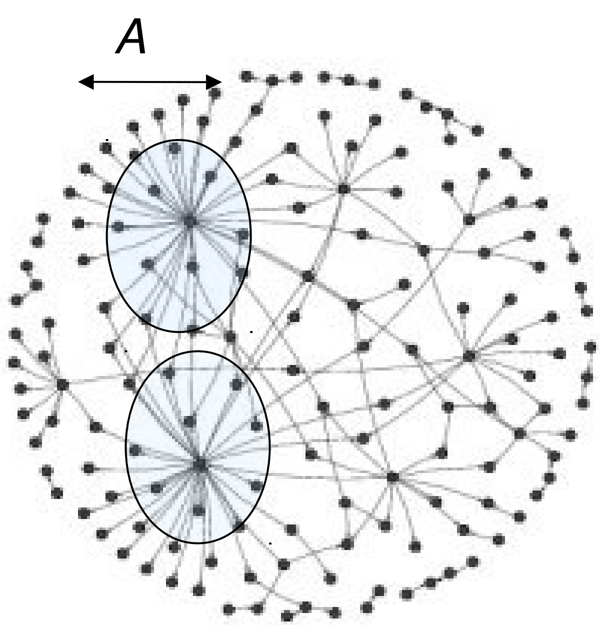
For SIS : Infections attempted for all nn

Order parameter : density of active (●) sites
*Regular, Euclidean lattice: **DP** critical point : $\lambda_c > 0$ between inactive and active phases*



Rare region effects in networks ?

Rare active regions below λ_c with: $\tau(A) \sim e^A$
→ slow dynamics (Griffiths Phase) ?



$$\rho(t) \sim \int dA_R \; A_R \; p(A_R) \; \exp[-t/\tau]$$

M. A. Munoz, R. Juhász, C. Castellano and G. Ódor, PRL 105, 128701

1. Inherent disorder in couplings
2. Disorder induced by topology

Optimal fluctuation theory + simulations: YES

- In weighted Erdős-Rényi random networks
- In networks with finite topological dimension

Quenched Mean-Field (QMF) theory for SIS

Rate equation of SIS for occupancy prob. at site i:

$$\frac{d\rho_i(t)}{dt} = -\rho_i(t) + \lambda(1-\rho_i(t))\sum_{j=1}^{N} A_{ij}w_{ij}\rho_j(t)$$

Weighted (real symmetric) Adjacency matrix: $B_{ij} = A_{ij}w_{ij},$

For $t \to \infty$ the system evolves into a steady state, with the probabilities expressed as

$$\rho_i = \frac{\lambda \sum_j B_{ij}\rho_j}{1+\lambda\sum_j B_{ij}\rho_j} . \quad (5)$$

Express ρ_i on orthonormal eigenvector ($\boldsymbol{f_i(\Lambda)}$ *) basis:*

$$\rho_i = \sum_{\Lambda} c(\Lambda) f_i(\Lambda). \quad (6)$$

Mean-field critical point estimate $\lambda_c = 1/\Lambda_1$

Total infection density vanishes near λ_c as :

$$\rho(\lambda) \approx \alpha_1\tau + \alpha_2\tau^2 + \ldots , \quad (8)$$

where $\tau = \lambda\Lambda_1 - 1 \ll 1$ with the coefficients

$$\alpha_j = \sum_{i=1}^{N} f_i(\Lambda_j)/[N\sum_{i=1}^{N} f_i^3(\Lambda_j)]. \quad (9)$$

To describe the localization of the components of $\boldsymbol{f}(\Lambda_1)$ [19] used the inverse participation ratio

$$IPR(\Lambda) \equiv \sum_{i=1}^{N} f_i^4(\Lambda), \quad (10)$$

QMF results for SIS on Erdős-Rényi graphs

Percolating ER

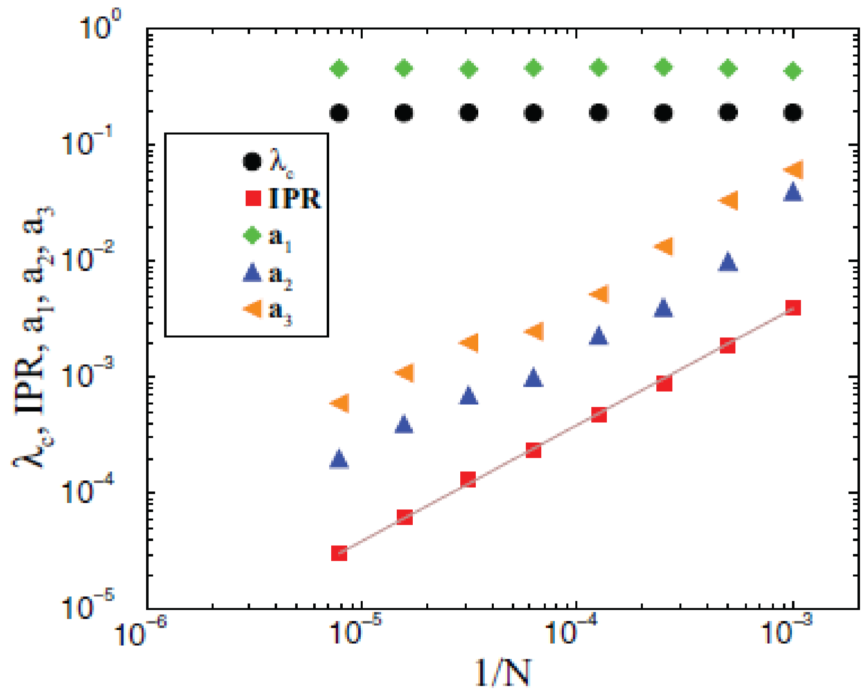


FIG. 1. (Color online) Finite size scaling of QMF results on the ER model with $\langle k \rangle = 4$ for $N = 1000, 2000, 4000, \ldots, 128\,000$. Bullets, λ_c; boxes, IPR; up-triangles, a_1; down-triangles, a_2; right-triangles, a_3. (Line) Least-squares fitting with the form $\sim 1/N$.

Fragmented ER

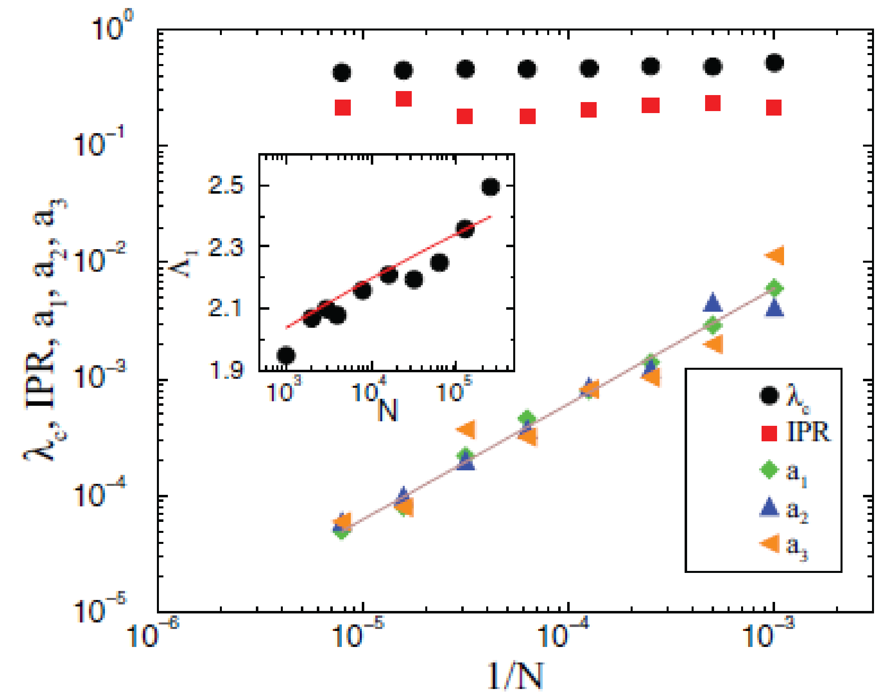


FIG. 3. (Color online) Finite size scaling of QMF results for ER graph with $\langle k \rangle = 0.3 N = 1000, 2000, 4000, \ldots 128\,000$. Bullets, λ_c; boxes, IPR; up-triangles, a_1; down-triangles, a_2; right-triangles, a_3. (Line) Least-squares fitting with a form $1/N$. (Inset) $\Lambda_1(N)$ (bullets) and the form (8) (line).

IPR ~ 1/N → delocalization

$\Lambda_1 = 1/\lambda_c \rightarrow 5.2(2) \;\leftrightarrow\; \Lambda_1 = \langle k \rangle = 4$

$a_1 >> a_2, a_3, \ldots$

IPR → 0.22(2) → localization

$$\Lambda_1(N) = \sqrt{\ln N / \ln \ln N}.$$

$a_1 \sim a_2 \sim a_3$: strong corrections to scaling,

Density decay simulation results on ER graphs

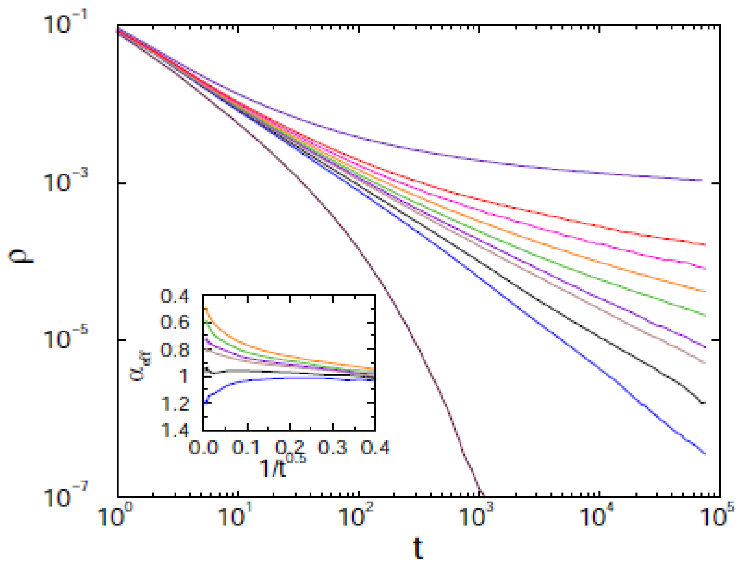


FIG. 2: (Color online) Density decay as a function of time for the SIS on ER graph with $\langle k \rangle = 1$. Network size $N = 10^6$. Different curves correspond to $\lambda = 1.05, 1.09, 1.095, 1.1, 1.02, 1.05, 1.095, 1.115, 1.12, 1.15$ (from bottom to top curves). Inset: effective exponents of the same data near the phase transition point.

Percolating ER

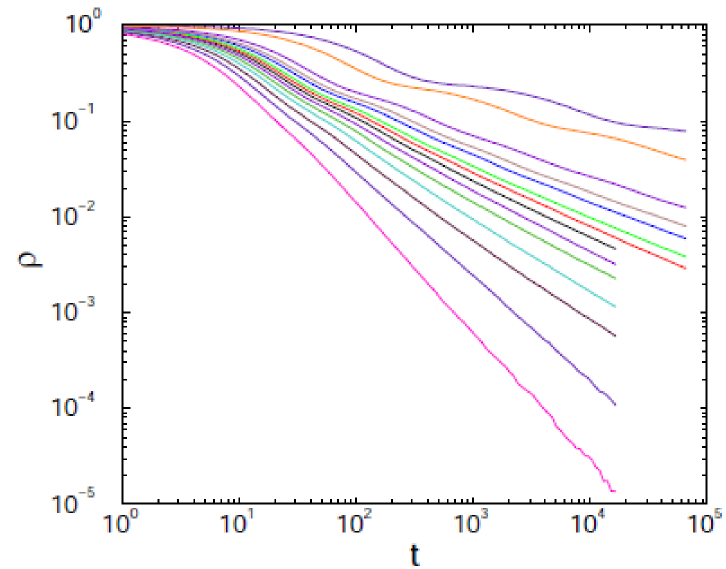


FIG. 4: (Color online) Density decay as a function of time in the SIS defined on ER graph with $\langle k \rangle = 0.3$. Network size $N = 10^6$. Different curves correspond to $\lambda = 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 20, 50, 100$ (from bottom to top curves).

Fragmented ER

Quenched Mean-Field for SIS on scale-free BA graph

Barabási-Albert graph lin. attachment prob.:

$$P_{s \to s'} = k_{s'} / \sum_{s'' < s} k_{s''}$$

$P(k) \sim k^{-3}$ $\qquad$ $\Lambda_1(N) \sim N^{1/4}$

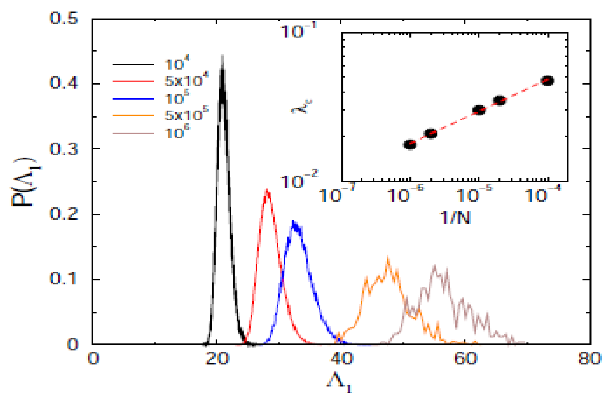


FIG. 6: (Color online) Probability distribution of IPR of the $m = 3$ BA SIS model for sizes $N = 10^4$, 5×10^4 10^5, 5×10^5 and $N = 10^6$ (from left to right).

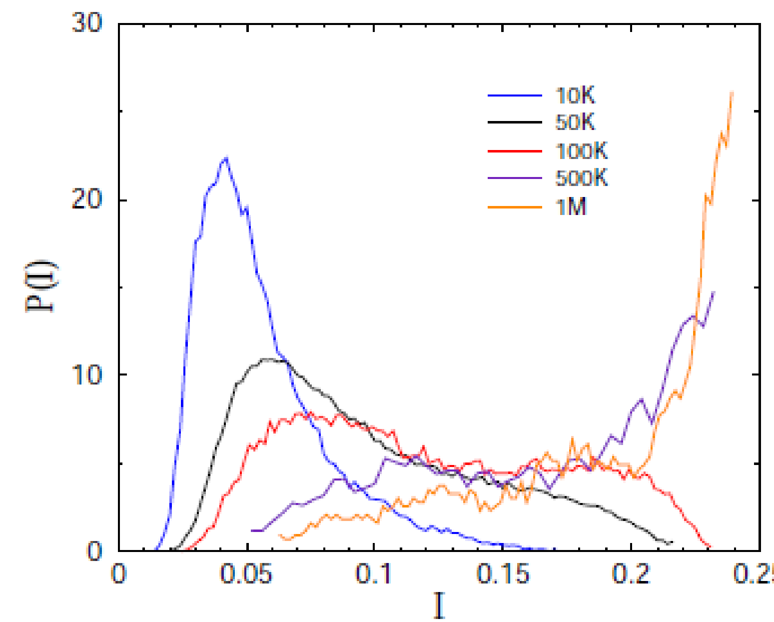


FIG. 6: (Color online) Probability distribution of IPR of the $m = 3$ BA SIS model for sizes $N = 10^4$, 5×10^4 10^5, 5×10^5 and $N = 10^6$ (from left to right).

IPR exhibit wide distributions
Localization for N → ∞

Rare-region effects in aging BA graphs

BA followed by preferential edge removal

$p_{ij}\ \mu\ k_i k_j$

dilution is repeated until 20% of links are removed

QMF

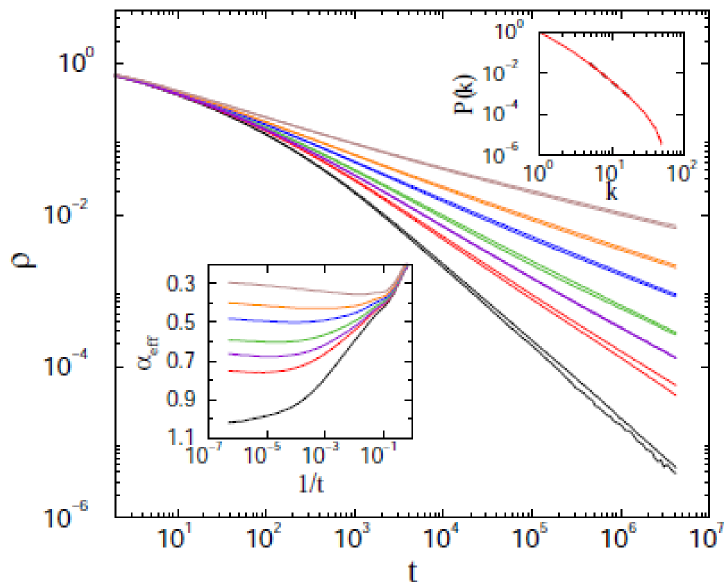


FIG. 7: (Color online) Density decay as a function of time for the SIS on BA graph with network sizes: $N = 10^5$ (thin lines), $N = 10^6$ (thick lines). Different curves correspond to $\lambda =$ 2.4, 2.45, 2.47, 2.5, 2.55, 2.6, 2.7 (from bottom to top curves). Left inset: Local slopes of the same data. Right inset: Degree distribution of the aging BA graph for $N = 4 \times 10^6$ nodes.

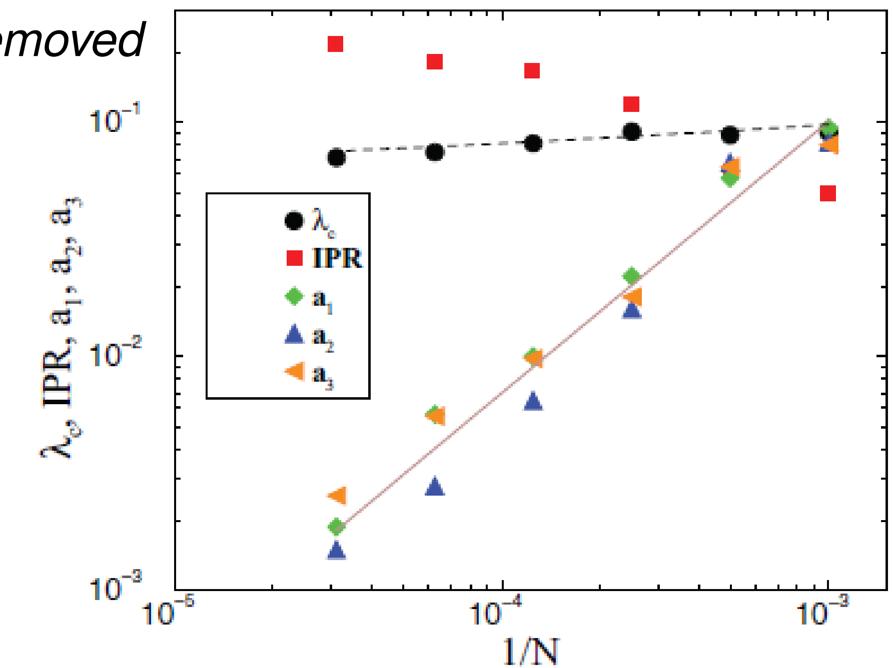


FIG. 6. (Color online) Finite size scaling of QMF SD results of the SIS on generalized BAT with $N = 1000, 2000, 4000, 8000, 16\,000, 32\,000$ nodes. Bullets, λ_c; boxes, IPR; up-triangles, a_1; down-triangles, a_2; right-triangles, a_3. (Line) Least-squares fitting with the form $\sim 1/N$.

$$\lambda_c = 1/\Lambda_1 = 0.001 + 0.17(1/N)^{0.18}$$

IPR → 0.28(5)

No size dependence → Griffiths Phase

Localization in the steady state

Bursts in the SIS model in one-dimension

- Density decay and seed simulations of SIS on a ring at the critical point
- Power-law inter-event time (Δ) distribution among subsequent infection attempts
- Invariance on the time window and initial conditions
- The tail distribution decays as the known auto-correlation function for $t \to \infty$

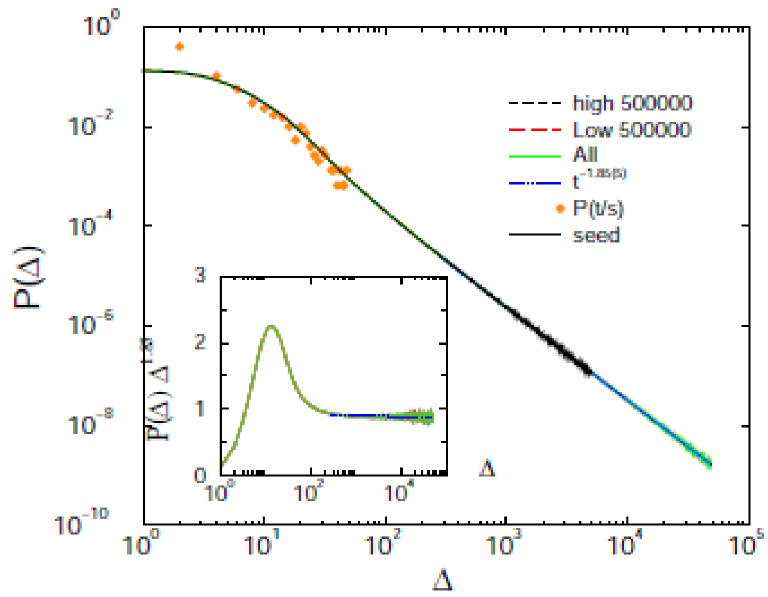


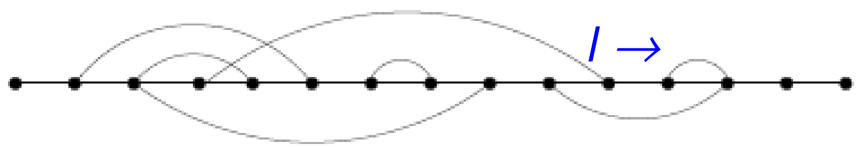
FIG. 1: (Color online) Inter-communication time distribution in the 1d critical CP of size $N = 10^5$. Full line denote histogramming from all times, dashed line from high times, long dashes from low times. The dotted line shows a power-law fit for $t > 200$ resulting in $\propto t^{-1.85(5)}$. The solid thin line corresponds to runs from seed initial conditions. Inset: the same data multiplied by the $t^{1.85}$ corresponding to the tail decay.

$$P(\Delta) \simeq p^2 \Gamma(t,s)$$

$$\Gamma(t,s) = \langle n_i(t) n_i(s) \rangle - \langle n_i(t) \rangle \langle n_i(s) \rangle$$

$$\Gamma(t,s) \propto (t/s)^{-\theta} = (\Delta/s + 1)^{-\theta}$$

Bursts of the SIS in Generalized Small World networks



$P(l) \sim \beta\, l^{-2}$

- Density decay and seed simulations of SIS on a near the critical point
- Power-law inter-event time (Δ) distribution among subsequent infection attempts
- Invariance on the time window and initial conditions
- The tail distribution decays with a λ dependent exponent around the critical point **in the extended Griffiths Phase**

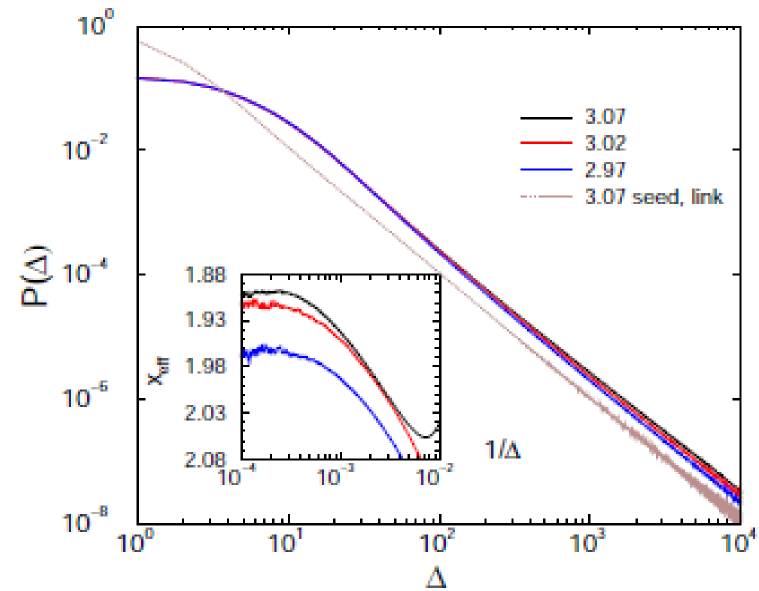


FIG. 2: (Color online) Inter-communication time distribution in the GP of a GSW network with $\beta = 0.1$ of size $N = 10^6$ and $\lambda = 2.97, 3.02, 3.07$ (bottom to top solid curves). The solid thin line corresponds to runs from seed initial conditions measuring all activation attempts. Inset: Effective exponents defined as (4) of the same data.

Bursts of the SIS in aging scale-free networks

- Barabasi-Albert model with preferential detachment of aging links: $P(k) \sim k^{-3} \exp(-ak)$
- Density decay simulations of SIS on a near the critical point
- Power-law inter-event time (Δ) distribution among subsequent infection attempts
- The tail distribution decays with a λ dependent exponent around the critical point in an **extended Griffiths Phase**

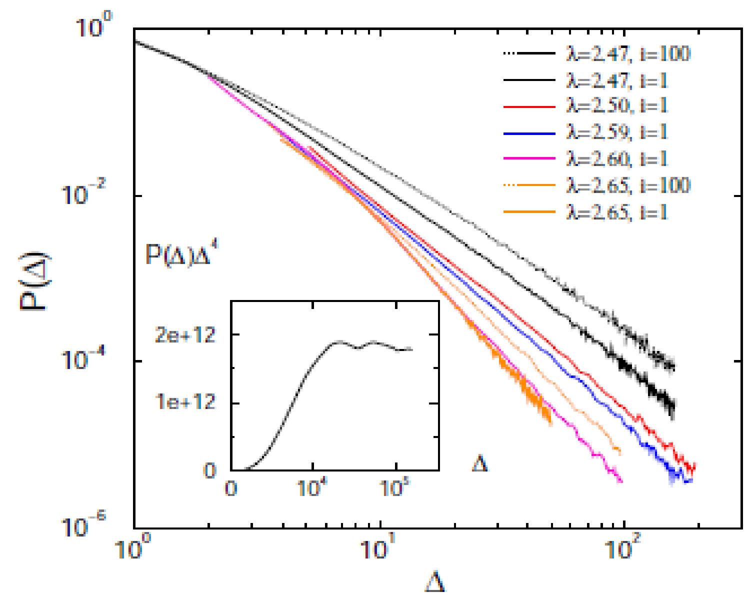


FIG. 4: (Color online) Inter-communication time distribution in the GP of an ageing BA network of size $N = 10^5$ for $\lambda = 2.47, 2.5, 2.55, 2.59, 2.6, 2.65$ (top to bottom curves) and measured at different ($i = 1$ and $i = 100$) sites. Power-law fitting exponents for the tail behavior is shown in the text. Inset: $P(\Delta)\Delta^4$ at the λ_c of the CP defined on the pure BA graph.

Summary

- ***Quasi-static disorder*** *in complex networks can cause slow (PL) dynamics :*
  Rare-regions, Griffiths phases in extended regions $\rightarrow$ *without SOC mechanism !*
- ***GP*** *can occur due to purely topological disorder in finite dimensional network models*
- *In* ***infinite dimensional*** *networks (ER, BA)* ***mean-field transition*** *of SIS with logarithmic corrections (HMF, simulations, QMF), but localization for* $\gamma > 3$
- *In* ***weighted*** *BA* ***trees*** *non-universal, slow, power-law dynamics can occur for finite N, but in the N* $\rightarrow \infty$ *limit: saturation occurs*
- ***GP*** *is related to* ***localization*** *in the steady state in most cases*
- *Bursty behavior in extended* ***GP****-s as a consequence of heterogeneity* $+\ \tau \rightarrow \infty$
  Complexity induces non-Markovian behavior of the agents of the network

[1] M. A. Munoz, R. Juhász, C. Castellano, G. Ódor, Phys. Rev. Lett. 105 (2010) 128701
[2] R. Juhász, G. Ódor, C. Castellano, M. A. Munoz, Phys. Rev. E 85 (2012) 066125
[3] G. Ódor, R. Pastor-Satorras, Phys. Rev. E 86 (2012) 026117
[4] G. Ódor, Phys. Rev. E 87, 042132 (2013)
[5] G. Ódor, Phys. Rev. E 88, 032109 (2013)
[6] G. Ódor, Phys. Rev. E 89, 042102 (2014)
[7] G. Ódor, Phys. Rev. E 90, 032110 (2014)

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SIS on weigthed Barabási-Albert graphs

- *Excluding loops slows down the spreading* + *Weights:*

WBAT-II: *disassortative weight scheme*

$$\omega_{ij} = \frac{|i-j|^x}{N} \qquad k_i \propto (N/i)^{1/2}$$

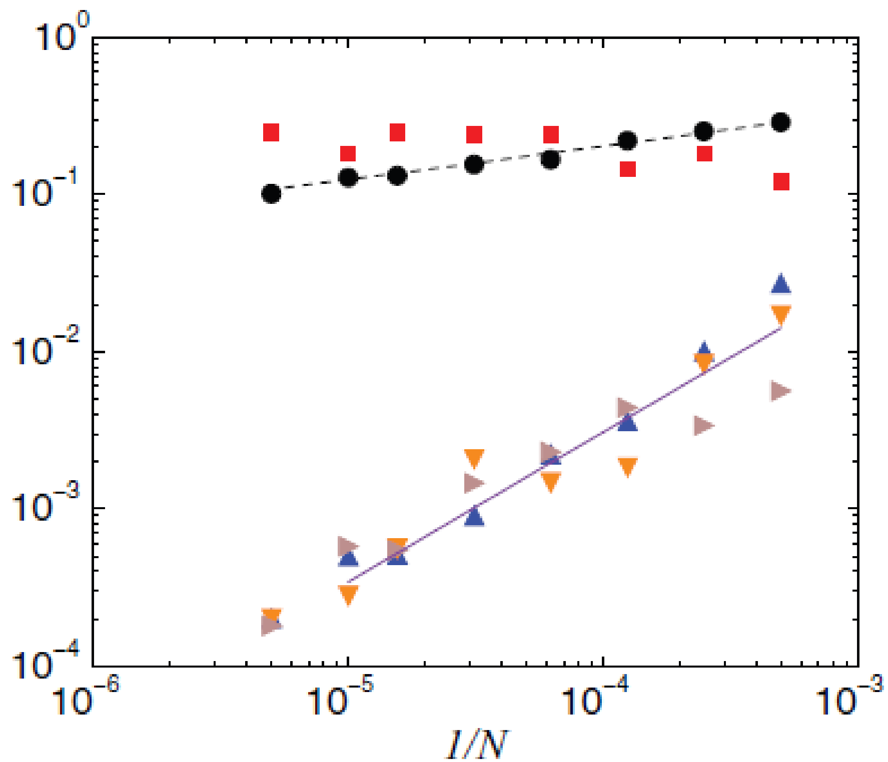


FIG. 2. (Color online) Finite-size scaling of QMF SD results for WBAT-II model for $N = 2\,000, 4\,000, \ldots 200\,000$. Bullets, λ_c; boxes, IPR; up-triangles, a_1; down-triangles, a_2; right-triangles, a_3. Lines: least-squares fitting for with the form Eq. (12).

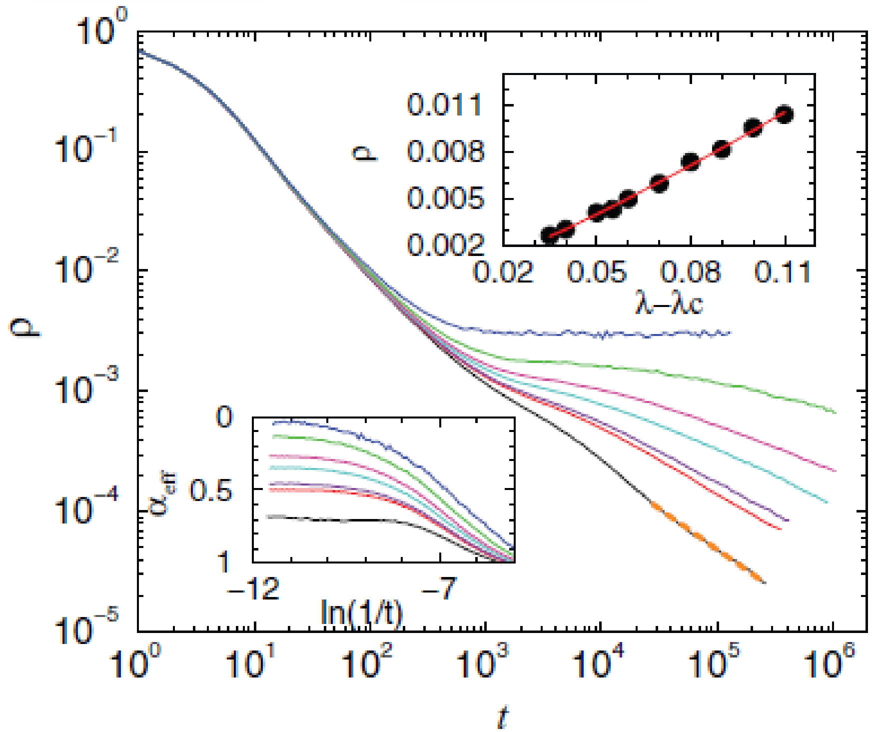
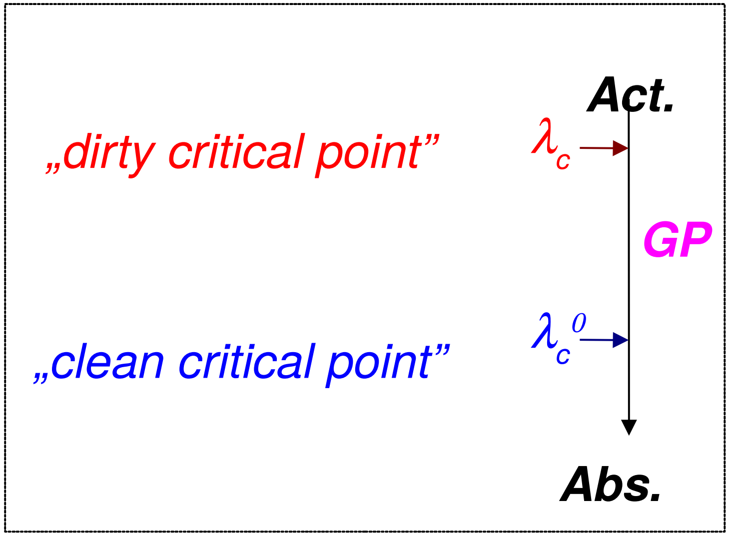


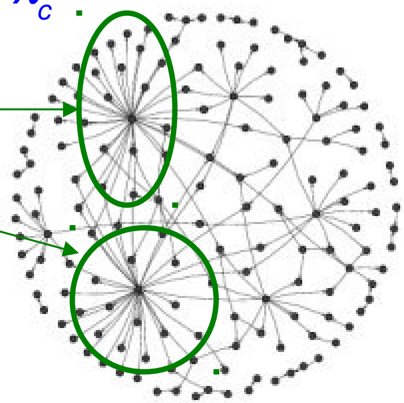
FIG. 4. (Color online) Density decay as a function of time for the SIS on weighted BA trees generated with the age-dependent (disassortative) WBAT-II scheme with exponent $x = 2$. Network size $N = 10^6$. Different curves correspond to $\lambda = 1.657$, 1.66, 1.661, 1.663, 1.67, 1.68, 1.69 (from bottom to top). Dashed line: power-law fit. Left inset: Local slopes of the same curves showing level-off for large times. Right inset: Steady-state density (bullets) above the epidemic threshold. The line shows power-law fitting with the form Eq. (14).

λ dependent density decay exponents:
Griffiths Phases or Smeared phase transition ?

Rare Region theory for quench disordered CP



- *Fixed (quenched) disorder/impurity* ***changes the local birth rate*** $\Rightarrow \lambda_c > \lambda_c^0$.
- ***Locally active**, but arbitrarily large Rare Regions in the inactive phase due to the inhomogeneities*



- *Probability of RR of size L_R:*

  $w(L_R) \sim \exp(-c\, L_R)$

  contribute to the density: $\rho(t) \sim \int dL_R\ L_R\ w(L_R)\ \exp[-t/\tau(L_R)]$
- ***For*** $\lambda < \lambda_c^0$ *: conventional (exponentially fast) decay*
- ***At*** λ_c^0 *the characteristic time scales as:* $\tau(L_R) \sim L_R^{\ Z} \Rightarrow$ *saddle point analysis:*

  $\ln \rho(t) \sim t^{d/(d+Z)}$ *stretched exponential*
- ***For*** $\lambda_c^0 < \lambda < \lambda_c$: $\tau(L_R) \sim \exp(b\, L_R)$: ***Griffiths Phase***

  $\rho(t) \sim t^{-c/b}$ *continuously changing exponents*
- ***At*** λ_c *: b may diverge* $\rightarrow$ $\rho(t) \sim \ln(t)^{-\alpha}$ *Infinite randomness fixed point scaling*
- ***In case of correlated RR-s with dimension*** $> d^-$ ***: smeared transition***